GRAVITATIONAL WAVES FROM TRI-AXIAL ROTATING STARS

An Analytical Boundary Value Problem Approach

Prakash Sarnobat – East Surrey Gravity Research, UK

info@esgravity.org.uk,   www.esgravity.org.uk

13th Belgian-Dutch Gravitational Wave Meeting, Nijmegen, October 2025.

About ‘East Surrey G.R.’

- Author got involved in a number of projects related to Analytical Models for Isolated Matter Distributions.
- Decided to bring them all ‘under one roof’; Institution set up specially for this. Represent UK in ICRANET (Ruffini).
- **Other Current Ongoing Projects include**:
- ‘Rotational/Tidal deformations beyond the Hartle-Thorne approximation’ (GRG24 Poster, Glasgow, July 2025).
- ‘An Exact Gravitational Wave from a bounded source’ (5th Scandinavian Meeting on GWs, Copenhagen, May 2025)
- ‘Bondi-Sachs perturbations of the Schwarzschild metric: Rotating Tri-axial ellipsoids’ (GraSP24 Poster, Pisa, 2024).

Relevant Background

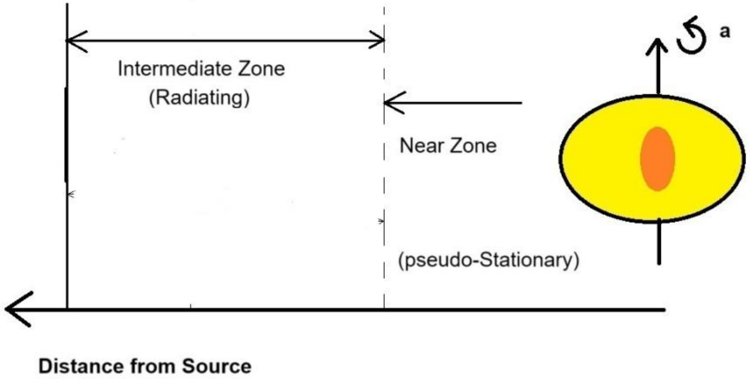
- Rotating Neutron Stars are a source of Continuous Waves, which can be used to infer properties of the star, e.g. EOS.
- Non-uniform deformation: Transient quadrupole moment.
- Incompressible interior is mathematically the simplest case
- Chandrasekhar (1967) – Post-newtonian Jacobi Ellipsoids for interior. Textbook examples use asymptotic vacuum.
- Blanchet-Damour (1990) – Near zone and Radiating zone
- Neugebauer-Meinel (2008) - Boundary value problem approach to stationary rotating stars
- Within the scope of **Linearized Gravity**, we can exploit the analogy with Electromagnetism to solve the equations.

Aims of this lecture

- Review of the ‘zonal’ decomposition, and its limitations.
- Recap the MacLaurin and Jacobi Ellipsoids, and attempt to re-frame them in the context of Linearized Gravity, for both interior and vacuum.
- Introduce the concept of a ‘small’ **non-uniform** deformation, and how it simplifies all the equations.
- Obtain the analytical form of the **radiating** external field.
- Describe the Boundary Value Problem approach.
- Compare with the Bondi-Sachs quasi-spherical Schwarzschild perturbation (*time-permitting*)

Zonal Decomposition

- Decompose the exterior region into a ‘Near Zone’, which is pseudo-stationary, and an intermediate zone, where radiation effects dominate (Blanchet-Damour, 1990).

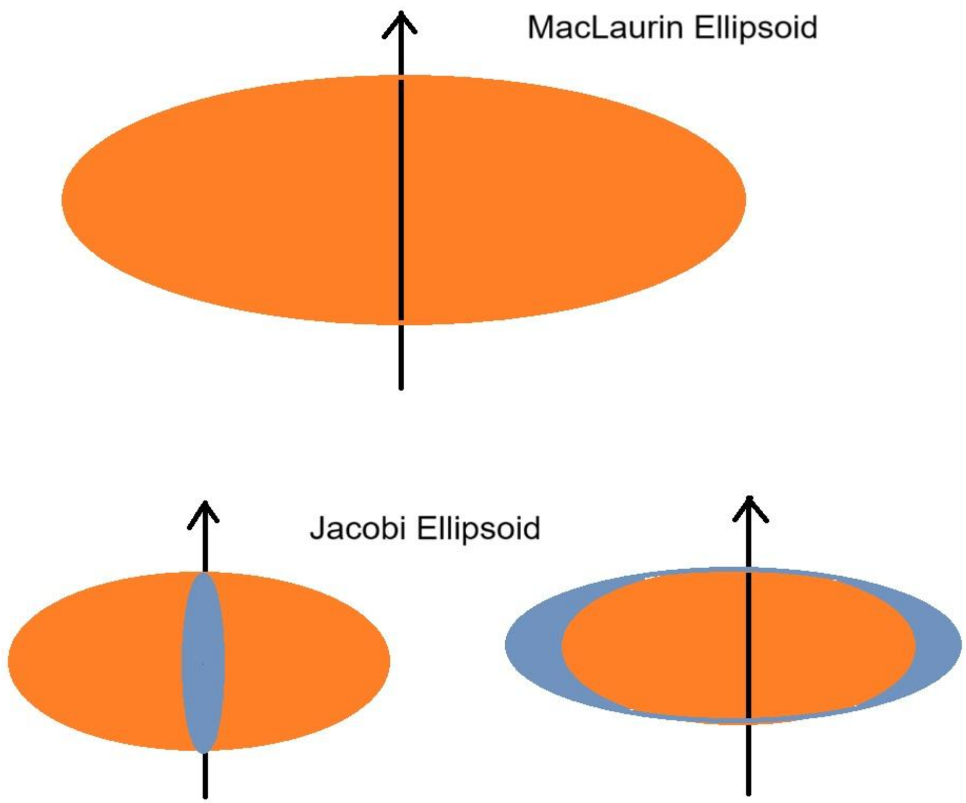


- But this decomposition is only valid if the emitted wavelength is larger than the size of the source. In the current work, we do **not** make this assumption!

MacLaurin and Jacobi Ellipsoids (1)

- Consider a ‘squashed ball’. Both equatorial axes have the same size, but the polar axis is different. Shorter=Oblate, Longer=Prolate.
- When viewed side-on, i.e. looking towards the equator, the shape of the body does not change during rotation, and therefore is stationary.
- But if the two equatorial axes are of *different* size, then when viewed from side-on, the shape does change during rotation, and is called a ‘Jacobi’ ellipsoid. In Electromagnetism, this would result in EM radiation.

MacLaurin and Jacobi Ellipsoids (2)



MacLaurin and Jacobi Ellipsoids (3)

- MacLaurin Ellipsoids are completely solved in Newtonian theory, for arbitrary rotation rates and flattening. Solution expressed in terms of Spheroidal Legendre functions.
- Neugebauer-Meinel (2008) demonstrated that by treating the setup as an interior-exterior **boundary value problem**, the Maclaurin axis-ratio relation arises without evaluating complicated integrals for the interior!
- Jacobi Ellipsoids cannot be solved in closed form; interior produces elliptic integrals, and exterior produces ellipsoidal harmonics. *New* approach required!
- Now reframe all this in Post-Minkowskian approximation.

Linearized Gravity

- To lowest order in $^1/_{c^2}$ get the decoupled EFEs:
- $\boxdot\, \bar{h}_{ab} = T_{ab}$ in the Harmonic Gauge
- Harmonic Gauge may be somewhat misleading!
- **OR**, in the Transverse Gauge (TT=Transverse Traceless, obtained from Projection Operator),
- $\triangle\, \Phi_N = \rho$ ('Coulomb' Poisson Equation)
- $\triangle\, h^{(TT)}{}_{0\gamma} = T^{(TT)}{}_{0\gamma}$ ('Solenoidal' Poisson Eqn)
- $\boxdot\, h^{(TT)}{}_{\beta\gamma} = T^{(TT)}{}_{\beta\gamma}$ (Wave Equation)

Helmholtz Equation (1)

- Fourier Transforming the Wave Equation leads to a Helmholtz type equation, which has *no* time-dependence.
- $\triangle H + kH = \tau$ (k = wavenumber)
- Mathematically speaking, the Poisson equation is a k=0 subcase of the Helmholtz equation.
- Previously, we also said that the spheroidal case is a stationary subcase of the tri-axial case. Therefore, the Helmholtz equation must reduce to the Laplace equation.
- Create a **single** solution procedure for the Helmholtz equation, with the Laplace solution procedure as subcase.

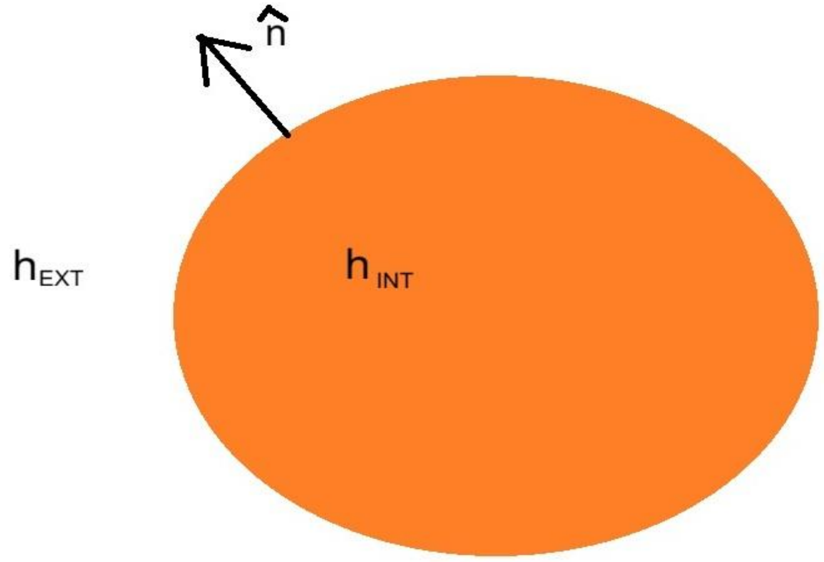
Helmholtz Equation (2)

- Focusing on the **vacuum** part for now, in tri-axial ellipsoidal coordinates the Helmholtz equation can be shown to admit a separation of variables.
- That separation would normally produce ellipsoidal harmonics, but in the spheroidal (here stationary) case, that same separation results in Legendre Polynomials.
- We now make the demand: If the two equatorial axes are only *slightly* different, can the solution be represented as:
- Stationary Legendre Polynomials + a radiative correction?

Perturbation of axes

- YES! Making that assumption about the axes allows the Laplacian operator to decompose in a similar manner.
- **Each** of the 3 resulting ODEs is of the form (a,b are foci):
- $(s^4 - s^2[a^2 + b^2] + a^2b^2)\frac{d^2F}{ds^2} - s(a^2 + b^2 - 2s^2)\frac{dF}{ds} = F(-mk^2s^4 - qs^2 - p)$
- **Ansatz**. Let b = a + m, and F = f(s) + m*h(s). m=0 simply gives Legendre Equation in oblate spheroidal coordinates.
- When the *first-order* equation is solved, the homogeneous part gives Legendre functions, and the inhomogeneous part gives a lengthy combination of rational functions and logs.

Boundary Conditions



- Darmois/Lichnerowicz conditions: At boundary, match
- $h_{EXT} = h_{INT}$
- $\partial_n h_{EXT} = \partial_n h_{INT}$

SUMMARY

- Reviewed the (classical) MacLaurin and Jacobi Ellipsoids, and their reformulation in Linearized GR.
- Argued that in order to get explicit closed-form solutions over the *whole* vacuum, a perturbation decomposition must be made pertaining to the two equatorial axes and foci.
- Used this decomposition to solve both the Laplace and Helmholtz equations. Solutions useful in both EM *and* GR.
- Still remaining – make a similar decomposition for the interior (with the mass density being constant), and apply boundary conditions of potentials and normal derivatives.
- Publications to follow!

THANK YOU!