HANDS-ON CONE DEFORMATIONS OF HYPERBOLIC MANIFOLDS

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GEOMETRY, DYNAMICS, AND ZETA FUNCTIONS
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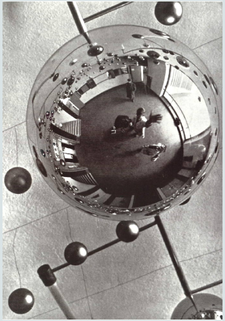


Walter Funkat (1929)

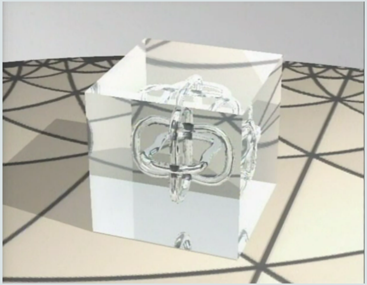
Definition

A hyperbolic 3-manifold is a Riemannian 3-manifold with a metric of constant curvature -1.

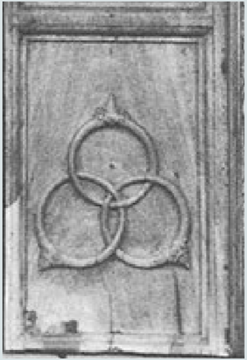
Every complete hyperbolic 3-manifold is of the form X/G where

- G is a discrete subgroup of $\mathrm{PSL}(2,\mathbb{C}) = \mathrm{Isom}^{+}(\mathbb{H}^3)$.
- $X \subseteq \mathbb{H}^3$ is preserved by G.

Example. The complement of the Borromean rings in $\mathbb{S}^3$.



C. Gunn, D. Maxwell *et al*, *Not Knot* (1991)



Door of the Church of San Sigismondo, Cremona. P. Cromwell. E. Beltrami, M. Rampichini. "The Borromean Rings". *Math. Intell.* (1998)

`https://aelzenaar.github.io/cones/borromean_rings_video.webm`

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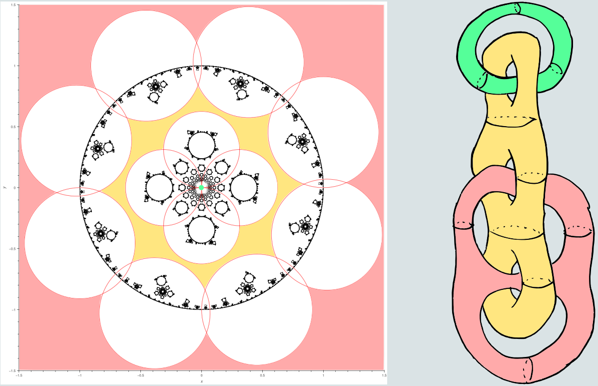
Example. A genus 3 compression body.



RIGIDITY AND IRRIGIDITY OF COMPLETE METRICS

Finite volume

Mostow–Prasad Rigidity Theorem. If M is a complete finite-volume hyperbolic 3-manifold then any discrete *isomorphic* deformation of the holonomy group in $\mathrm{PSL}(2, \mathbb{C})$ actually produces a discrete *conjugate* group.

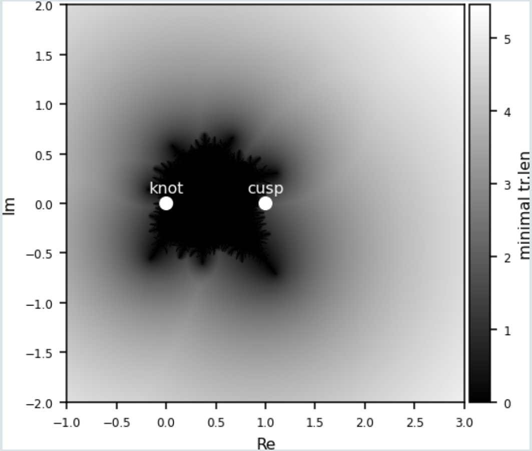
Equivalently, no finite volume complete hyperbolic 3-manifold admits metric deformations through complete hyperbolic metrics.

E.g., all link complements.

Infinite volume

An Ahlfors–Bers theorem. If M is a complete hyperbolic 3-manifold which flares to a conformal end S (in particular, it is infinite volume), then the set of deformations of M is parameterised by $\mathrm{Teich}(S)$.

E.g., all hyperbolic compression bodies.

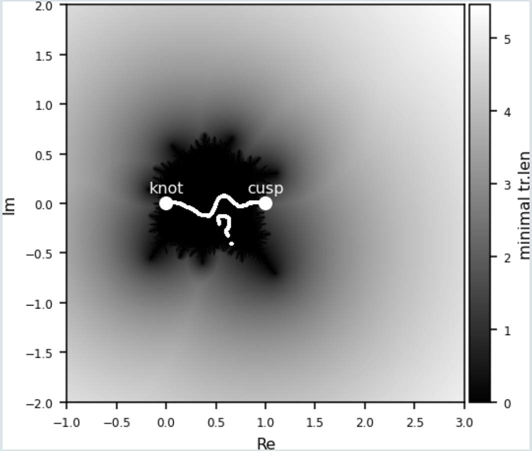


A linear slice through $\mathrm{Hom}((\mathbb{Z} \oplus \mathbb{Z}) * \mathbb{Z}, \mathrm{PSL}(2, \mathbb{C}))$.

E., "From disc patterns in the plane to character varieties of knot groups", arXiv:2503.13829 (to appear in a volume of *Contemp. Math.*)

Question. Can you interpolate smoothly between different components of discrete locus in the character variety via a smooth path of indiscrete groups that have meaningful geometry?

Can you deform a knot group (rigid) into a group lying in a Teichmüller space?



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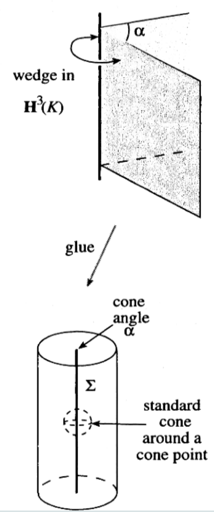
Lemma (Thurston, 1977). Every compact hyperbolic 3-manifold M with only torus boundary components admits a deformation through *incomplete* hyperbolic structures.

Proof in two parts.

1. *Every compact hyperbolic 3-manifold with only torus boundary components has a deformable metric*, by computing degrees of freedom of the holonomy representation $\pi_1(M) \to \mathrm{PSL}(2, \mathbb{C})$.
2. *Every such deformation is actually an incomplete hyperbolic structure*: algebraic deformation of matrices induces a deformation of the Dirichlet polyhedron (Voronoi tile) in $\mathbb{H}^3$. Ideal vertices might degenerate from a rank 2 parabolic group to a rank 2 elliptic-loxodromic group (indiscrete & modelling a cone arc).



p.61 of Cooper, Hodgson, Kerchkoff, *Three-dimensional orbifolds and cone-manifolds*, Math. Soc. Japan (2000).

Theorem (Hodgson and Kerchkoff, 1998). If there is a sufficently large radius embedded tube around a geodesic γ in a hyperbolic 3-manifold M, then there exists a family of deformations M_θ of the metric on $M - \gamma$ so that the completion of M_θ is a hyperbolic cone manifold with singular angle θ along the geodesic γ.

Proof sketch. Actually, Hodgson–Kerchkoff show that if the cone angle along γ is fixed then the hyperbolic structure is rigid. Since by Thurston's lemma there is always a cone deformation starting at M, it follows that this cone deformation must change the angle.

To show this rigidity, they observe that infinitesimal deformations of the metric determine Hodge harmonic elements of $H^1(M, E)$ (E = bundle of infinitesimal isometries of M). For a deformation which leaves the cone angle fixed, it is possible to find a Hodge harmonic representative with real symmetric part η such that $\|D\eta\|^2_{L^2(M)} + \|\eta\|^2_{L^2(M)} \to 0$ on the boundary of tubes bounding γ; thus $\eta = 0$.

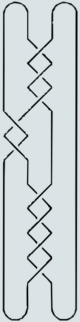
Problem with Hodgson–Kerckhoff

Need strong geometric control on the injectivity radius of the geodesic (even if you do not care about getting effective bounds out you need to put explicit bounds in).

Sometimes this can be achieved for large classes of examples, if you have enough hyperbolic data.

Example/Theorem (Purcell, 2008). Let K be a prime knot in $\mathbb{S}^3$ and choose a 'sufficiently simplified' diagram in the plane with a total of $n \geq 2$ twist regions, with each twist region containing at least $c \geq 116$ crossings. Then there is an effective lower bound on (approximately) cusp length, namely height $\geq n(1 - f(c))^2$ where $f(c) > 0$ and $f(c) \to 0$ as $c \to \infty$.

2**-bridge knots** are knots obtained by alternately braiding pairs of four strands together and capping the ends off:



M. Sakuma. “The Topology, Geometry and Algebra of Unknotting Tunnels”. *Chaos Solitons Fractals* (1998)

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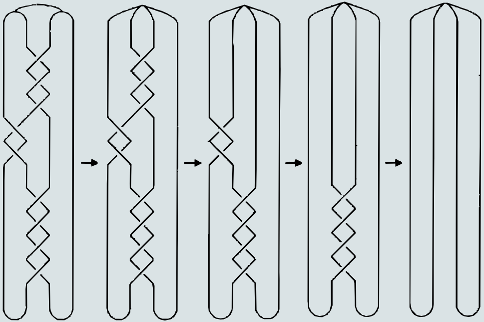


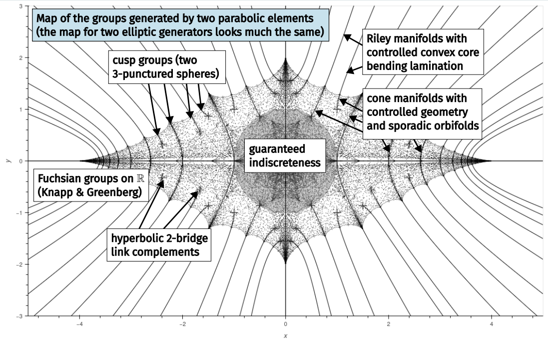
M. Sakuma. "The Topology, Geometry and Algebra of Unknotting Tunnels". *Chaos Solitons Fractals* (1998)

Unknotting tunnel for a knot k: A 1-cell τ so that $\mathbb{S}^3 - (k \cup \tau)$ is isotopic to an unknotted handlebody.

Conjecture

There is a continuous family of incomplete hyperbolic metrics interpolating between $\mathbb{S}^3 - (k \cup \tau)$ and $\mathbb{S}^3 - k$ given by cone-deforming τ from 0 to 2π.

H. Akiyoshi, M. Sakuma, M. Wada, & Y. Yamashita. *Punctured torus groups and 2-bridge knot groups, I.* Springer (2007).
D. Lee & M. Sakuma. "A variation of McShane's identity for 2-bridge links". *G&T* (2013).


Map of the groups generated by two parabolic elements (the map for two elliptic generators looks much the same)

cusp groups (two 3-punctured spheres)

Riley manifolds with controlled convex core bending lamination

cone manifolds with controlled geometry and sporadic orbifolds

guaranteed indiscreteness

Fuchsian groups on $\mathbb{R}$ (Knapp & Greenberg)

hyperbolic 2-bridge link complements

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Theorem (Conway, 1970).
Two-bridge links all arise from taking one pair of strands in a ball and alternating half-twists along the equator with half-twists along the prime meridian.

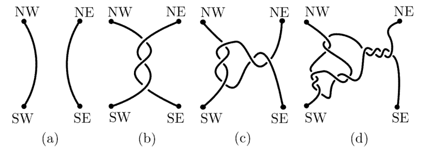
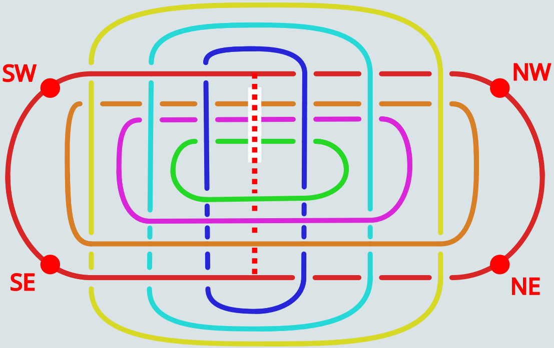


Figure 10.1. Building a rational tangle from the continued fraction $[4, -2, -2, 3]$.

J. Purcell. *Hyperbolic knot theory*. Amer. Math. Soc. (2021)

Strategy for a version of Sakuma’s conjecture.

1. Model twists and the associated braiding on strands in the ball by Dehn fillings.
2. Show that the cone deformation works before doing the Dehn filling.
3. Check that the Dehn filling does not affect the metric ‘far away’ from the tori boundary components being filled, so that the cone deformation continues to work even after Dehn filling.



Strategy for an attack on Sakuma's conjecture.

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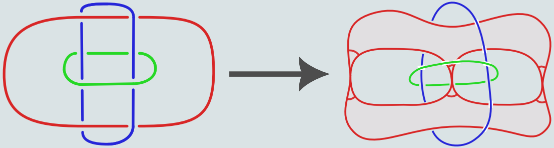
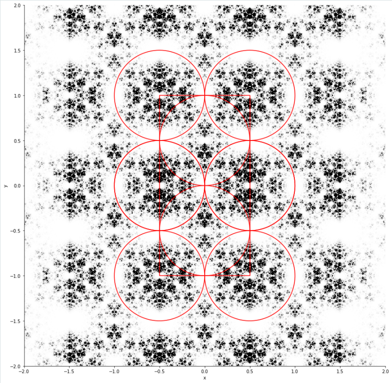
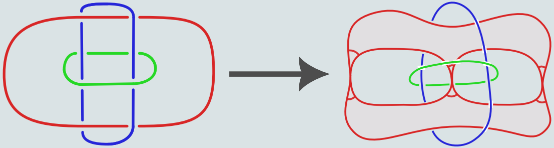
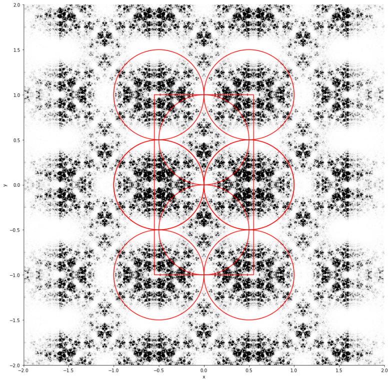




Theorem (E., 2025). There exists a cone deformation which drills out an arc from the Borromean rings via incomplete hyperbolic metrics.





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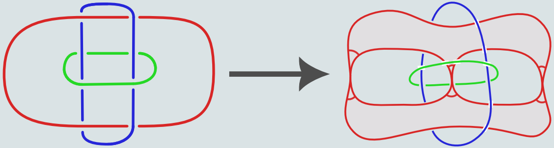

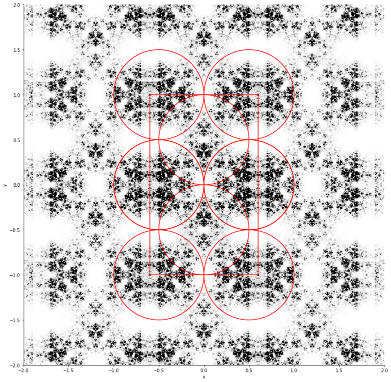

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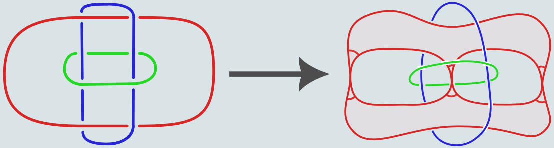

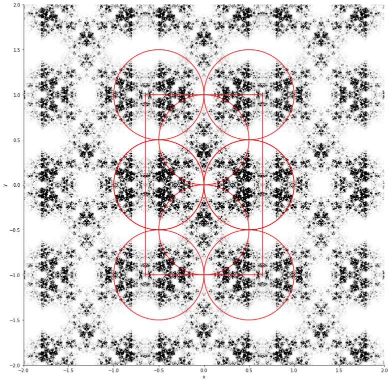

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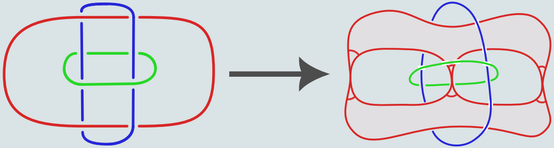

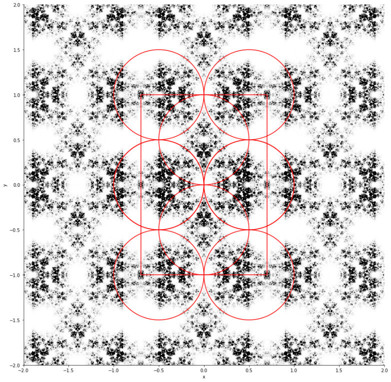

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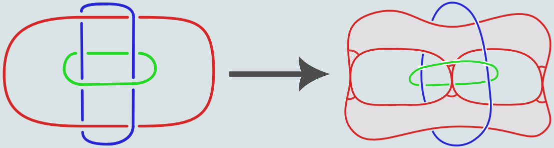

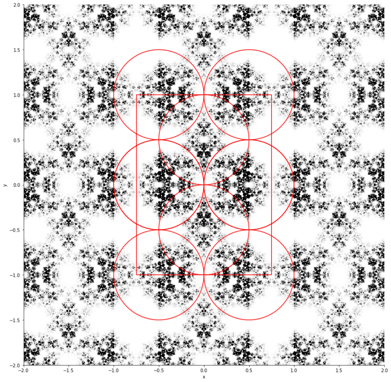

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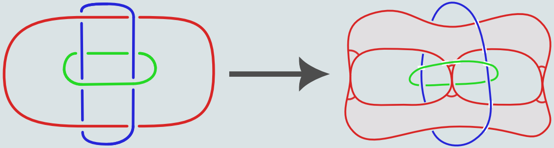

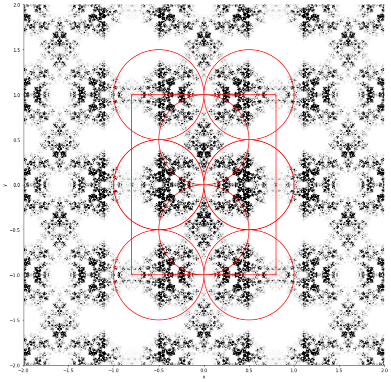

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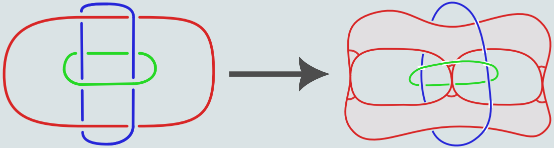

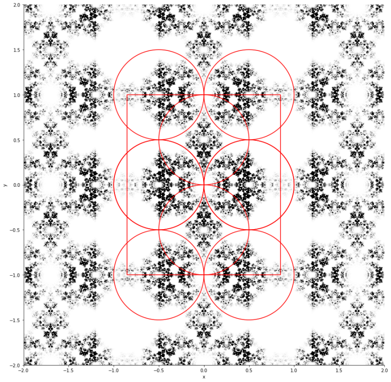

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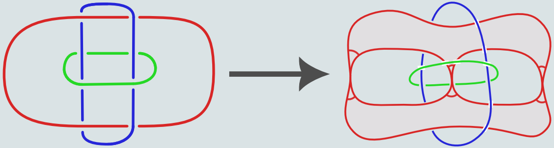

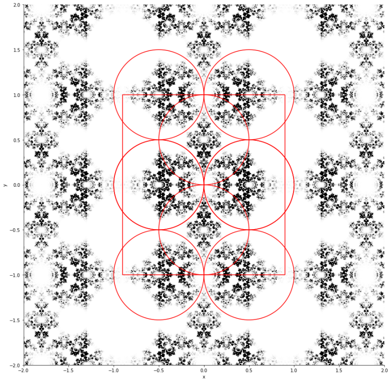

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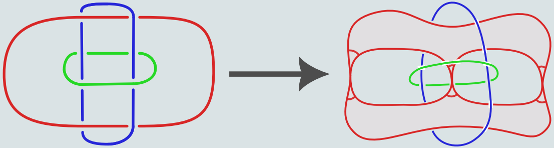

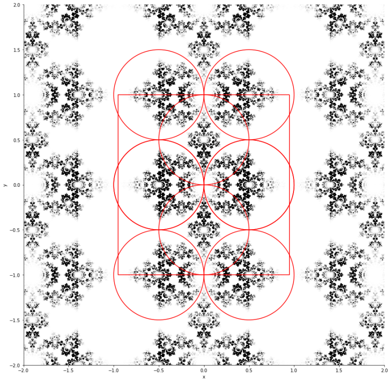

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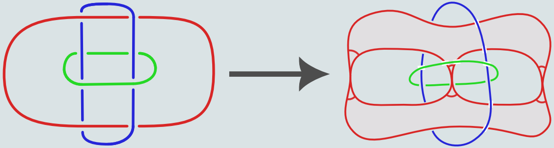

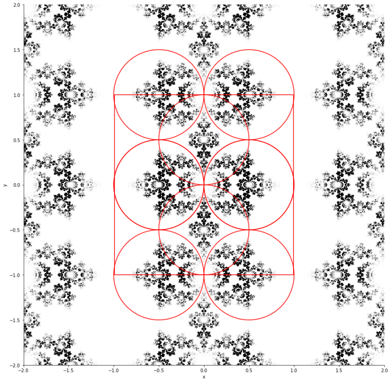

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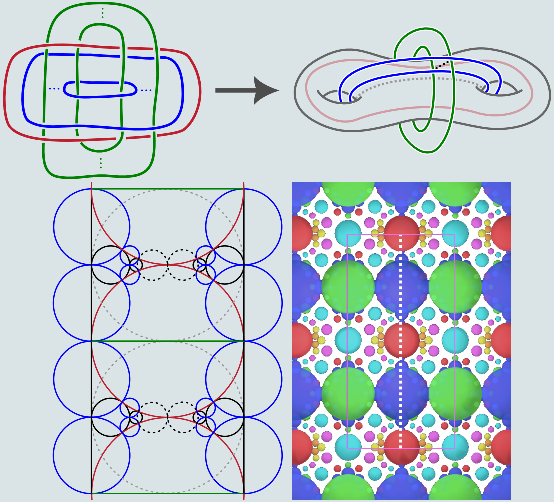


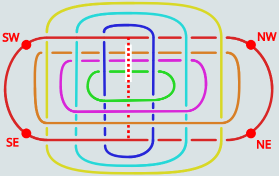
Theorem (E., 2025). There exists a cone deformation which drills out an arc from the n-stacked Borromean rings via incomplete hyperbolic metrics.

Proof. Draw a fundamental domain as on the left. The red circles correspond to circles paired by one of the parabolics preserving the red cusp horoball in the right picture. At infinity in this picture is a cusp other than the red one, and increasing the horizontal translation length produces an ideal cone arc as with the Borromean rings case.

2π Theorem (Gromov & Thurston; Bleiler & Hodgson, 1996).

Let V be a solid torus supplied with a hyperbolic metric near its boundary so that ∂V is the quotient of a horosphere. Then the metric near the boundary can be extended to a negatively curved metric on V provided that the length of the Euclidean geodesic representing the meridian curve on ∂V is at least 2π.

Proof. By explicit construction.



Theorem (E., 2025). If k is a 2-bridge link with continued fraction expansion $[2a_1, \ldots, 2a_n]$, all $a_i > 6$ except $a_1 > 4$, then there is a continuous deformation of a complete negatively curved metric on $\mathbb{S}^3 - k$ through incomplete negatively curved metrics which drills out the unknotting tunnel. Further, these negatively curved metrics have locally constant curvature -1 in a large neighbourhood of the tunnel.

1. The cone deformation through incomplete hyperbolic metrics works for a large infinite family of links, *fully augmented links*, and is an isometry far away from the axis being deformed (*geometric isolation*).
2. Since every hyperbolic 3-manifold is obtained by Dehn filling a fully augmented link, we get cone deformations through incomplete negatively curved metrics for a large family of 3-manifolds.
3. When Dehn filling using the 2π theorem, we obtain a negatively curved metric. By geometrisation (c.f. arXiv:math/0612069, p.354) there exists a hyperbolic metric on the complement of the cone arc. But this metric may have an entirely different angle around the cone arc.
4. **If M is a negatively curved 3-manifold with cone singularities, does M admit a hyperbolic metric away from the singularities that preserves the cone angles?**

SELECTED READING

- A. Elzenaar, “Expansion joints in hyperbolic manifolds”. arXiv:2512.00879
- W. P. Thurston, *The geometry and topology of three-manifolds.* lecture notes (c. 1979) + Amer. Math. Soc. (2022).
- C. D. Hodgson & S. P. Kerchkoff, “Rigidity of hyperbolic cone-manifolds and hyperbolic Dehn surgery”. *JDG* (1998).
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- D. Futer, J. S. Purcell, & S. Schleimer, “Effective bilipschitz bounds on drilling and filling”. *G&T* (2022).