

Spectral zeta function densities on asymptotically Minkowski spaces

joint works with

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Introduction

Laplace–Beltrami on **asymptotically Minkowski** (\mathbb{R}^n, g) :

$$\square_g = \partial_t^2 - \Delta_y + O(\langle x \rangle^{-\mu}).$$

Here $x = (t, y)$. Questions:

- ▶ For $\text{Im } z \neq 0$, solve

$$(\square_g - z)u = f, \quad f \in L^2(\mathbb{R}^n)$$

in $u \in L^2(\mathbb{R}^n)$? (**Self-adjointness**).

- ▶ What happens when $\text{Im } z \rightarrow 0^+$?
- ▶ What is $(\square_g + m^2)^{-\alpha}$ for $\alpha \in \mathbb{C}$, $m^2 \geq 0$, trace density?
- ▶ Make sense of **quantum stress-energy tensor**?

$$\hat{T}_{\mu\nu}(x) = -\frac{1}{\sqrt{\det g(x)}} \frac{\delta}{\delta g^{\mu\nu}(x)} (\text{Tr} \log(\square_g + m^2)).$$

Looks like we want not usual hyperbolic PDE theory, but something more analogous to asymptotically Euclidean Δ_g .

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The key are **Feynman asymptotic conditions**.

Feynman asymptotic conditions

Suppose $\square_g = \partial_t^2 - \Delta$, $\text{Im } z > 0$. Retarded propagator of $\square_g - z$:

$$\theta(t-s) \frac{e^{i(t-s)\sqrt{-\Delta-z}} - e^{-i(t-s)\sqrt{-\Delta-z}}}{2i\sqrt{-\Delta-z}}$$

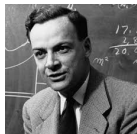
This is **not compatible with** $\|(\square_g - z)^{-1}\| \leq |\text{Im } z|^{-1}$. But:

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This is **not compatible with** $\|(\square_g - z)^{-1}\| \leq |\text{Im } z|^{-1}$. But:



“Every particle in Nature has an amplitude to move backwards in time, and therefore has an **anti-particle**.”

– Richard Feynman

$$((\square_g - z)^{-1}u)(t, \cdot) = -\frac{1}{2} \int \frac{e^{-i|t-s|\sqrt{-\Delta-z}}}{\sqrt{-\Delta-z}} u(s, \cdot) ds. \quad (1)$$

The boundary value $(\square_g + m^2 - i0)^{-1}$ is the **Feynman propagator**.

But for general \square_g with t -dependent coefficients, nothing like (1) exists...

💡 Start with (1) at **infinity**, then propagate!

Suggests the following for $\text{Im } z \geq 0$:

$u \in H^s(\mathbb{R}^n)$ satisfies **Feynman conditions** if $\exists u_{\pm\mp} \in H^s(\mathbb{R}^{n-1})$ s.t.

$$u(t) = \begin{cases} e^{-it\sqrt{-\Delta-z}}u_{+-} + \text{l.o.t. for } t \gg 1 \\ e^{it\sqrt{-\Delta-z}}u_{-+} + \text{l.o.t. for } t \ll -1 \end{cases}$$

(reversing signs defines **anti-Feynman** conditions)

Microlocally, distinguishes components Σ_{\pm} of $\sigma_{\text{pr}}(\square_g)^{-1}(\{0\})$ at ∞ .

Feynman conditions vs. geometric scattering:

- Gell-Redman–Haber–Vasy '16: consider $\text{Im } z = 0$, $\text{Re } z = 0$, non-trapping, short-range case $\mu > 1$, and use **microlocal propagation+radial estimates** at ∞ in spirit of Melrose '94 and Vasy '13.
- Gérard–Wrochna '19–'20: for $\text{Im } z = 0$, $\text{Re } z > 0$, **invertibility** of $\square_z - z$ on “lossy graph norm” Sobolev spaces with **Feynman conditions**.
- Vasy '20, Nakamura–Taira '21–'22: **essential self-adjointness** of \square_g , and **limiting absorption principle**.

Spectral zeta functions and geometry

Dang–Wrochna '23

For $\varepsilon > 0$, the trace density $(\square_g - i\varepsilon)^{-\alpha}(x, x)$ exists as meromorphic function of $\alpha \in \mathbb{C}$. Furthermore,

$$\lim_{\varepsilon \rightarrow 0^+} \operatorname{res}_{\alpha = \frac{n}{2} - 1} (\square_g - i\varepsilon)^{-\alpha}(x, x) = \frac{R_g(x)}{i6(4\pi)^{\frac{n}{2}} \Gamma(\frac{n}{2} - 1)},$$

where $R_g(x) = \text{scalar curvature at } x \in M$.

\Rightarrow Hence **non-linear Einstein equations** from spectral ζ -function density.

Lorentzian analogue of local version of spectral ζ -function $\operatorname{Tr}_{L^2}(\Delta_g^{-\alpha})$.

Dang–Wrochna '21

Poles of $\zeta_{g,\varepsilon}(\alpha) := (\square_g - i\varepsilon)^{-\alpha}(x, x)$ can be recovered by scalings.
(generalizes **Guillemin–Wodzicki residue** of Ψ DOs)

Scaling towards the diagonal

Let $\Delta = \{(x, x) \mid x \in M\}$.

A vector field X is **radial** (or **Euler**) if $Xf = f$ modulo quadratically vanishing terms for all f with $f|_{\Delta} = 0$.

Locally there are coordinates $(x^i, h^i)_{i=1}^n$ s.t. $\Delta = \{h^i = 0\}$ and $X = \sum_{i=1}^n h^i \partial_{h^i}$.

$u \in \mathcal{D}'_{\Gamma}(\mathcal{U})$ is **log-polyhomogeneous** if

$$e^{-tX}u = \sum_{p \leq k \leq N, 0 \leq i \leq l-1} e^{-tk} \frac{(-1)^i t^i}{i!} (X - k)^i u_k + \mathcal{O}_{\mathcal{D}'_{\Gamma}(\mathcal{U})}(e^{-t(N+1-\varepsilon)}).$$

Pollicott–Ruelle resonances of the flow e^{-tX} are the poles of

$$\int_0^{\infty} e^{-tz} \langle (e^{-tX}u), \varphi \rangle dt = \sum_{k=p, 0 \leq i \leq l-1}^N (-1)^i \frac{\langle (X - k)^i u_k, \varphi \rangle}{(z + k)^{i+1}}$$

+ holomorphic on $\operatorname{Re} z \leq N$.

Dynamical definition of residue

Suppose $\Gamma|_{\Delta} \subset N^* \Delta$. Let $\Pi_0 :=$ projection on zero resonance.

The **dynamical residue of \mathcal{K}** (w.r.t. X) is:

$$\text{res}_X \mathcal{K} = \iota_{\Delta}^* (X(\Pi_0(\mathcal{K}))) \in C^{\infty}(M).$$

Might be ill-defined, and might depend on X . But...

Dang–Wrochna '21

For all radial X and all $k = 1, \dots, \frac{n}{2}$ and $\varepsilon > 0$,

$$\text{res}_{\alpha=k} \zeta_{g,\varepsilon}(\alpha) = \frac{1}{2} \text{res}_X ((\square_g - i\varepsilon)^{-k}),$$

where $\zeta_{g,\varepsilon}(\alpha)$ is the **spectral zeta function density** of $\square_g - i\varepsilon$.

“Analytic residues of $\zeta_{g,\varepsilon}$ are **dynamical residues** (scaling anomalies).”

Spectral zeta functions and geometry

Dang–Wrochna '23

For $\varepsilon > 0$, the trace density $(\square_g - i\varepsilon)^{-\alpha}(x, x)$ exists as meromorphic function of $\alpha \in \mathbb{C}$. Furthermore,

$$\lim_{\varepsilon \rightarrow 0^+} \operatorname{res}_{\alpha = \frac{n}{2} - 1} (\square_g - i\varepsilon)^{-\alpha}(x, x) = \frac{R_g(x)}{i6(4\pi)^{\frac{n}{2}} \Gamma(\frac{n}{2} - 1)},$$

where $R_g(x)$ is the scalar curvature at $x \in M$.

\Rightarrow Hence **non-linear Einstein equations** from spectral ζ -function density.

The key is the **special structure of singularities** of $(\square_g - z)^{-1}(x, y)$ (**Feynman wavefront set**) \Rightarrow nice composition properties.

Think of $(x \pm i0)^{-1}$ on \mathbb{R} : product $(x + i0)^{-1}(x - i0)^{-1}$ is problematic but $(x - i0)^{-1}(x - i0)^{-1} = (x - i0)^{-2}$ is OK.

Note: For $(\square_g - i\varepsilon)^{-\alpha}$, good control of microlocal estimates as $\operatorname{Im} z \gg 0$ needed.

■ odd dimensions, arbitrary signature of g : work in progress (Kyae)

⚠ We want tensorial wave operators (squared twisted Dirac operator \mathbb{D}^2)

Remarks on self-adjointness


Back to **asymptotically Minkowski** $P = \square_g$, for self-adjointness we want

$$(u \in L^2(\mathbb{R}^n), (P \pm i)u = 0) \implies u = 0$$


Naive argument $\pm 2i \|u\|_{L^2}^2 = \langle Pu, u \rangle_{L^2} - \langle u, Pu \rangle_{L^2}$ reduces to

$$(u \in L^2(\mathbb{R}^n), (P \pm i)u = 0) \implies u \in H_{sc}^{\frac{1}{2}, -\frac{1}{2}}(\mathbb{R}^n).$$

Vasy '20 uses radial/propagation estimates in $\Psi_{sc}^{s, \ell}(\mathbb{R}^n)$ (cf. **Melrose '94**)

-  For now this gives **only** $H_{sc}^{\frac{1}{2}, -\frac{1}{2} - \epsilon}(\mathbb{R}^n)$... Similar (though weaker) conclusions from **Chihara '02** smoothing estimates (cf. **Taira '21**):

$$\|e^{-isP}u\|_{L^2([-T, T], H^{\frac{1}{2}, -\frac{1}{2} - \epsilon})} \leq C \|u\|_{L^2}$$

-  For squared **Dirac** operator formal self-adjointness **not true** (cannot use borderline estimate of **Vasy '20**)

Main result

Suppose $P = \square_g$ on (M, g) **small perturbation of Minkowski space**, or more generally, **tensorial** wave operator $P \in \Psi_{sc}^{2,0}$ close to non-trapping dynamics and

$$P^* - P \in \Psi_{sc}^{1,-1}(M; V)$$

in auxiliary $L^2(M; V)$ space.

Dang–Vasy–Wrochna '24

Then, $(P - z)^{-1}$ exists for $|\operatorname{Im} z| \gg 0$, and:

$$\operatorname{sp}(P) \subset \mathbb{R} \cup \{\text{some isolated poles in } |\operatorname{Im} z| \leq R\}$$

Furthermore, **large $|\operatorname{Im} z|$ microlocal resolvent estimates** in resolved weighted Sobolev spaces + *complex powers, spectral action, gauge fields, etc.*

(applies e.g. to squared Dirac operator $P = \not{D}^2$ despite **non-selfadjointness**)

For **scalar** $P = P^* = \square_g$, generalization to **radiative spacetimes** due to **Jia–Molodyk–Sussman '26**. Work in progress on $(P - z)^{-1}$ for $|z| \gg 0$.

Generalization: radiative spacetimes

In the scattering/asymptotically Minkowski setting the coefficients are smooth (or symbolic) at the radial compactification. For metrics produced by nonlinear waves this is too rigid: the natural asymptotics are **radiative**.

Blow up the boundary of the light cone

$$S = \{R = v = 0\}, \quad R = \langle x \rangle^{-1}, \quad v = \frac{t - |y|}{\langle x \rangle},$$

obtaining $[M; S]$ and the new front face \mathcal{I} (null infinity).

- Coefficients/solutions are **conormal** on $[M; S]$ rather than smooth sc-symbols on M (with logarithmic features in $3 + 1$ dimensions).
- The Hamilton flow has **new radial and saddle structure** at \mathcal{I} ; propagation/radial estimates move to de-sc framework **Sussmann '26**.
- **Dang–Molodyk–Vasy–W. (work in progress)**: large spectral parameter estimates for $(P - \lambda/h^2)^{-1} \Rightarrow$ complex powers, local invariants.

Positive commutator estimates (mock version)

Toy model: $P = P^*$ bounded, and \exists bounded A and D s.t.:

$$[P, iA] \geq (\mathbf{1} + D^2)^s. \quad (*)$$

Undo the commutator:

$$\begin{aligned} \frac{1}{2} \langle [P, iA]u, u \rangle &= \frac{\langle APu, u \rangle - \langle PAu, u \rangle}{2i} \\ &= \frac{\langle Pu, Au \rangle - \langle Au, Pu \rangle}{2i} \leq |\langle Pu, Au \rangle|, \end{aligned}$$

By Cauchy–Schwarz,

$$|\langle Pu, Au \rangle| \leq C \|(\mathbf{1} + D^2)^{-s/2} Pu\| \|(\mathbf{1} + D^2)^{s/2} u\| =: C \|Pu\|_{-s} \|u\|_s.$$

In combination with (*):

$$\|u\|_s^2 \leq C \|Pu\|_{-s} \|u\|_s,$$

hence invertibility statement $\|u\|_s \leq C \|Pu\|_{-s}$.

Positive commutator estimates

The existence of suitable A s.t.

$$[P, iA] \geq (\mathbf{1} + D^2)^s.$$

is **not expected**. But we can expect to prove it “somewhere in phase space”.

- ▶ If $P \in \Psi^s(M)$ and $A \in \Psi^\ell(M)$ then $[P, iA] \in \Psi^{s+\ell-1}(M)$ and

$$\sigma_{\text{pr}}([P, iA]) = \{p, a\} \bmod S^{s+\ell-2}(M).$$

The flow of $\{p, \cdot\}$ in $\{p = 0\}$ is the classical Hamilton flow, or **bicharacteristic flow** (note that in $\{p \neq 0\}$ elliptic theory applies).

- ▶ non-compact settings require **weighted Sobolev spaces**: extra weight $(\mathbf{1} + |x|^2)^\ell$ ($\Psi_{\text{sc}}^{m,\ell}(M)$ calculus)
- ▶ non-selfadjointness can be serious trouble (if we know nothing of $P - P^*$), or valuable help (for instance $P - i\varepsilon$ with $\varepsilon > 0$)
- ▶ **sub-principal symbol** of $P - z$ matters: $(P - z)A - A(P^* - \bar{z})$

From null bicharacteristic flow to global estimates

Flat metric $g_0 = dx_0^2 - (dx_1^2 + \cdots + dx_{n-1}^2)$ on \mathbb{R}^n extends to **radial compactification** $M = \overline{\mathbb{R}^n}$ defined using boundary-defining function $\rho = (x_0^2 + x_1^2 + \cdots + x_{n-1}^2)^{-\frac{1}{2}}$. Regularity w.r.t. $\rho^2 \partial_\rho = -\partial_r$

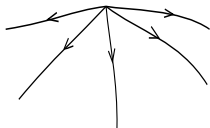
Lorentzian sc-metrics are C^∞ sections of ${}^{\text{sc}}T^*M \otimes_s {}^{\text{sc}}T^*M$, where ${}^{\text{sc}}T^*M$ generated by $\rho^{-2}d\rho, \rho^{-1}dy_1, \dots, \rho^{-1}dy_{n-1}$.

(Rescaled) null geodesics lift to **null bicharacteristics** on $\overline{{}^{\text{sc}}T^*M}$.

In $\Psi_{\text{sc}}^{2,0}(M)$, $\square_g - z$ has a “usual” principal symbol ($z \in \mathbb{C}$ plays no role) and a principal symbol at ∂M (**elliptic** for $\text{Im } z > 0$)

(M, g) **non-trapping** if there are sinks/sources L_\pm at ∞ , and null bicharacteristics flow from and to L_- and L_+ .

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dynamics of null bicharacteristics in $\overline{scT^*M}$



classical quantities increasing along flow



pos. commutator estimates in $\Psi_{sc}^{m,\ell}$ -calculus

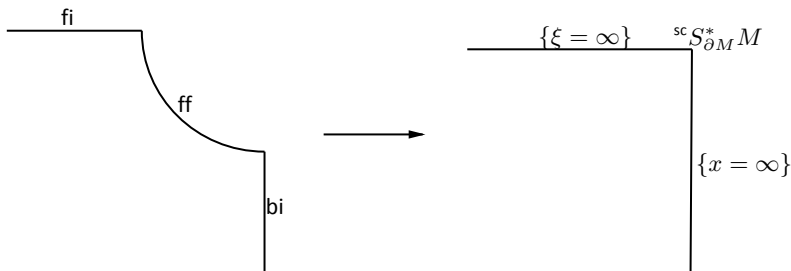


Get **Fredholm property** on anisotropic weighted $H_{sc}^{s,\ell}(M)$ graph norm spaces + (needs extra arguments) **invertibility** of $P-z$ for $|\operatorname{Im} z| \gg 0$, then use **analytic Fredholm theory**

Crucially, **threshold** conditions $\ell|_{L_+} < -\frac{1}{2}$ and $\ell|_{L_-} > -\frac{1}{2}$ (so $\ell = -\frac{1}{2}$ not allowed!).

Microlocalisation on blow-up of $\overline{{}^{\text{sc}}T^*M}$

Blow-up $[\overline{{}^{\text{sc}}T^*M}; {}^{\text{sc}}S_{\partial M}^*M]$ of the corner ${}^{\text{sc}}S_{\partial M}^*M$:



Resolved calculus $\Psi_{\text{res}}^{m,k,\ell}$ = microlocally $\Psi_{\text{sc}}^{k-\ell,\ell}$ near bi and $\Psi_{\text{sc}}^{m,k-m}$ near fi

1. Gives fully microlocal versions of [Taira '21](#) subelliptic estimate ($\text{Im } z \neq 0$):

$$u \in H_{\text{sc}}^{m+\frac{1}{2},\ell-\frac{1}{2}}, (P-z)u \in H_{\text{sc}}^{m,\ell} \Rightarrow u \in H_{\text{sc}}^{m,\ell}.$$

2. For $P-z$ with $\text{Im } z \neq 0$, z becomes sub-elliptic at ff.
3. Microlocalizers involving functions of $\langle \xi \rangle (\langle x \rangle + \langle \xi \rangle)^{-1}$, $\langle x \rangle (\langle \xi \rangle + \langle x \rangle)^{-1}$
4. Flexible threshold conditions $\pm(k-m)|_{L_{\mp}} > -\frac{1}{2}$.

Mixed classical–semiclassical resolution of $\overline{\text{sc}T^*}M \times [0, 1]_h$

1. Instead of $\text{Im } z \gg 0$, think of $h \rightarrow 0^+$ for

$$P - \lambda/h^2 = h^{-2}(h^2P - \lambda) =: h^{-2}(P_{\hbar} - \lambda)$$

2. **Semi-classical** operators

$$\text{Op}_{\hbar}(a)u(z) = \frac{1}{(2\pi)^n} \int e^{i(z-z') \cdot \zeta_{\hbar}/h} a(z, \zeta_{\hbar}) u(z') dz' d\zeta_{\hbar}.$$

can be seen as $\text{Op}(a(z, \zeta_{\hbar}/h))$, but problematic h^{-1} when differentiated!

3. Blow-up of $\zeta_{\hbar} = \infty$, $h = 0$ gives

$$\Psi_{\text{qsc,sc},\hbar^2,\hbar,\text{cl}}^{m,k,\ell,p,q,r}(M).$$

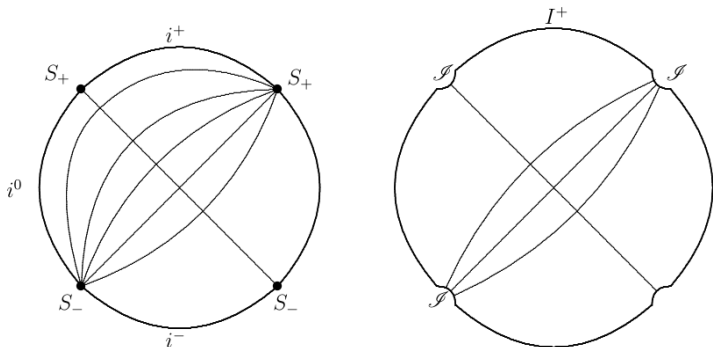
4. Elliptic/propagation/radial estimates give eventually estimates like

$$\|u\|_{s,k,\ell,p,q,r} \lesssim \|(P - \lambda/h^2)u\|_{s-1,k,\ell,p-2,q-2,r-2}$$

This **interpolates between gain in $\text{Im } \lambda/h^2$ and regularity**. Think

$$\|(-\Delta - z)^{-1}\|_{B(H^s, H^s)} = O(|\text{Im } z|^{-1}) \text{ vs. } \|(-\Delta - z)^{-1}\|_{B(H^s, H^{s+2})} = O(1).$$

The base blow-up at null infinity



Left: usual radial compactification, with sources/sinks over the boundary of the light cone. *Right:* after blowing up S , the front face(s) \mathcal{I} record radiation at null infinity; old sc source/sink picture is replaced by radial/saddle analysis at null infinity.

To sum up...

Feynman asymptotic conditions and non-elliptic Fredholm theory for \square_g are “closest possible” to asymptotically Euclidean \triangle_g .

Adding small $i\varepsilon$ or even better, $\text{Im } z \gg 0$, helps.

Positive commutator estimates in classical-semiclassical resolution of ${}^{\text{sc}}T^*M \times [0, 1)_h$ give the whole package: optimal propagation estimates, radial estimates, invertibility for $\text{Im } z \gg 0$, z -dependent wavefront set of $(P - z)^{-1}$, etc.

Related developments/questions:

- ▶ Lorentzian index theory Bär–Strohmaier '19-'23, Shen–Wrochna '22, ...
Feynman conditions \leftrightarrow Atiyah–Patodi–Singer conditions
- ▶ results on asymptotically de Sitter Frahm–Spilioti '23, Zeitoun '25, and asymptotically static Nakamura–Taira '23
- ▶ black hole space-times? Schwarzschild and other spacetimes with trapping?
- ▶ ζ -renormalized quantum $T_{\mu\nu}$ for Einstein equations via $\text{Tr } \log(\square_g)$, $H_{\text{sc}}^{s,\ell}$ estimates

Thank you for your attention!

Appendix

Spectral zeta function

(M, g) compact Riemannian $\implies \Delta_g$ has discrete spectrum.

Recall Riemann zeta $\zeta(\alpha) = \sum_{\lambda=1}^{\infty} \lambda^{-\alpha}$, then **spectral zeta**:

$$\mathbb{C} \ni \alpha \mapsto \zeta_{\Delta}(\alpha) = \sum_{\lambda \in \text{sp}(\Delta_g) \setminus \{0\}} \lambda^{-\alpha}.$$

Theorem (Minakshisundaram–Pleijel, Seeley)

The function $\zeta_{\Delta}(\alpha) = \text{Tr}_{L^2}(\Delta_g^{-\alpha})$ is **holomorphic** on $\text{Re } \alpha > \frac{n}{2}$, with **meromorphic continuation** to $\alpha \in \mathbb{C}$ and **poles** at $\{\frac{n}{2}, \frac{n}{2} - 1, \dots, 1\}$.

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+local version with densities:

$\alpha \mapsto \Delta_g^{-\alpha}(x, x)$ **holomorphic** on $\text{Re } \alpha > \frac{n}{2}$, with **meromorphic continuation** to $\alpha \in \mathbb{C}$ and poles at $\{\frac{n}{2}, \frac{n}{2} - 1, \dots, 1\}$, **smooth** in $x \in M$.

Here $\Delta_g^{-\alpha}(x, x')$ is the **Schwartz kernel** of $\Delta^{-\alpha}$, so

$$\text{Tr}_{L^2}(\Delta_g^{-\alpha}) = \int_M \Delta_g^{-\alpha}(x, x) dx$$

Example: \square_g on Minkowski space

Let $\Delta = \sum_{i=1}^{n-1} \partial_{x_i}^2$. The operator $\square_g = \partial_t^2 - \Delta$ is **essentially self-adjoint** on $L^2(\mathbb{R}^n)$. For $\text{Im } z > 0$, the resolvent is:

$$((\square_g - z)^{-1}u)(t, \cdot) = -\frac{1}{2} \int \frac{e^{-i|t-s|\sqrt{-\Delta-z}}}{\sqrt{-\Delta-z}} u(s, \cdot) ds.$$

The boundary value $\lim_{\varepsilon \rightarrow 0} (\square_g - i\varepsilon)^{-1}$ is called **Feynman inverse**.

It is the inverse of \square_g on weighted L^2 spaces with **non-local boundary conditions**. Indeed, \square_g is equivalent to

$$D = \partial_t - iA, \quad A = \begin{pmatrix} 0 & 1 \\ -\Delta & 0 \end{pmatrix},$$

and the Feynman inverse becomes

$$D^{-1}(t, s) = e^{i(t-s)A} \times \begin{cases} \mathbf{1}_{\mathbb{R}_+}(A) & \text{if } t > s \\ -\mathbf{1}_{\mathbb{R}_-}(A) & \text{if } t < s. \end{cases}$$

So D^{-1} maps to **Ker $\mathbf{1}_{\mathbb{R}_-}(A)$** for large $+t$ and to **Ker $\mathbf{1}_{\mathbb{R}_+}(A)$** for large $-t$.

Dirac operator

Suppose $SM \rightarrow M$ is a complex spinor bundle. It is endowed with a linear map

$$\gamma : C^\infty(M; TM) \rightarrow C^\infty(M; \text{End}(SM))$$

called **Clifford multiplication** and satisfying

$$\gamma(X)\gamma(Y) + \gamma(Y)\gamma(X) = -2(X \cdot gY)\mathbf{1}, \quad X, Y \in C^\infty(M; TM).$$

Furthermore, one is given a connection ∇^{SM} on SM , called **spin connection**, s.t. for all $X, Y \in C^\infty(M; TM)$ and $\psi \in C^\infty(M; SM)$,

$$\nabla_X^{SM}(\gamma(Y)\psi) = \gamma(\nabla_X Y)\psi + \gamma(Y)\nabla_X^{SM}\psi.$$

The **Lorentzian Dirac operator** \not{D} in a local frame (e_0, e_1, \dots, e_d) of TM is:

$$\not{D} = g^{\mu\nu} \gamma(e_\mu) \nabla_{e_\nu}^{SM}, \quad \not{D}^2 = \square_g + \text{l.o.t.}$$

on $C^\infty(M; SM)$. But unlike in Riemannian case, it is **not** formally self-adjoint.

Main result

Let $\gamma_0 := \gamma(e_0)$. If (M, g) is Minkowski space, then $\mathbb{D} = \gamma_0(i\partial_t - H)$ where $H^* = H$.

If (M, g) small perturbation of Minkowski space, $P := \mathbb{D}^2$ satisfies

$$P^* - P \in \Psi_{sc}^{1, -1-\delta}(M), \quad \delta > 0$$

for the scalar product $\langle \cdot, \gamma_0 \cdot \rangle_{L^2(M; SM)}$ used in quantization.

Theorem (Dang, Vasy, Wrochna)

$P := \mathbb{D}^2$ is a closed operator, and:

$$\text{sp}(P) \subset \mathbb{R} \cup \{\text{some isolated poles in } |\text{Im } z| \leq R\}$$

Furthermore, *large $|\text{Im } z|$ microlocal resolvent estimates*, spectral ζ function density, etc.

Spectral action principle

Assume $n = \dim M$ is even.

Theorem (Dang, Vasy, Wrochna)

For all $\varepsilon > 0$, the Schwartz kernel of $(P - i\varepsilon)^{-\alpha}$ has for $\operatorname{Re} \alpha > \frac{n}{2}$ a well-defined on-diagonal restriction $(P - i\varepsilon)^{-\alpha}(x, x)$, which extends as a meromorphic function with poles at $\{\frac{n}{2}, \frac{n}{2} - 1, \frac{n}{2} - 2, \dots, 1\}$.

Furthermore, as $\varepsilon \rightarrow 0^+$,

$$\operatorname{res}_{\alpha=\frac{n}{2}-1} \operatorname{tr}_E \left((P - i\varepsilon)^{-\alpha} \right) (x, x) \rightarrow \frac{\operatorname{rk}(E) R_g(x)}{i6(4\pi)^{\frac{n}{2}} \Gamma\left(\frac{n}{2} - 1\right)} + \frac{2\operatorname{tr}_E(F^E)(x)}{i(4\pi)^{\frac{n}{2}} \Gamma\left(\frac{n}{2} - 1\right)},$$

where R_g is the scalar curvature of (M, g) and F^E is the twisting curvature of $E \rightarrow M$.

Alternative formulation

Theorem (Dang, Wrochna '20)

For any Schwartz f with Fourier transform in $]0, +\infty[$,

$$f((\square_g + i\varepsilon)/\lambda^2)(x, x) = \sum_{j=0}^N \lambda^{n-2j} C_j(f) a_j(x) + \mathcal{O}(\varepsilon, \lambda^{n-2N-1}),$$

where $a_j(x)$ are Hadamard coefficients.