

# Approximation of $L^2$ -invariants of locally symmetric spaces

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# 1. Spectral invariants of compact manifolds

- ▶  $X$  compact Riemannian manifold,  $\dim X = n$ .
- ▶ Hodge – de Rham Thm: Betti numbers  $b_p(X) = \dim \mathcal{H}^p(X)$ .
- ▶  $\chi(X)$ ,  $\text{Sign}(X)$ ,
- ▶  $D: C^\infty(X, E) \rightarrow C^\infty(X, F)$  elliptic differential operator,  $\text{Ind}(D)$ .
- ▶  $A: C^\infty(X, E) \rightarrow C^\infty(X, E)$  Dirac type operator,  $\eta(A)$  eta invariant.
- ▶  $\rho: \pi_1(X) \rightarrow \text{GL}(V)$  representation,  $T_X(\rho)$  analytic torsion.

## **Analytic torsion:**

$E \rightarrow X$  flat vector bundle, associated to  $\rho$ ,  $h$  Hermitian fibre metric in  $E$ .

- ▶  $\Delta_p(\rho): \Lambda^p(X; E) \rightarrow \Lambda^p(X; E)$  Laplace operator on  $E$ -valued  $p$ -forms.
- ▶  $\text{Spec}(\Delta_p(\rho)): \lambda_1 \leq \lambda_2 \leq \dots \rightarrow \infty$ .
- ▶  $\zeta_p(s, \rho) = \sum_{\lambda_j > 0} \lambda_j^{-s}$ ,  $\text{Re}(s) > n/2$ .

$$\det \Delta_p(\rho) := \exp \left( - \frac{d}{ds} \zeta_p(s; \rho) \Big|_{s=0} \right).$$

$$T_X(\rho) = \prod_{p=1}^n [\det \Delta_p(\rho)]^{(-1)^{p+1} p/2}.$$

## 2. $L^2$ -invariants

Atiyah, 1976:  $L^2$ -index theorem,  $L^2$ -Betti numbers,

Lott, Mathai, 1992:  $L^2$ -torsion, Lück, 2004 :  $L^2$ -invariants.

- ▶  $\tilde{X} \rightarrow X$  universal covering of  $X$ ,  $\Gamma := \pi_1(X)$ ,  $d\tilde{\mu}$   $\Gamma$ -invariant measure on  $\tilde{X}$ .
- ▶ **Assume:**  $\tilde{X}$  is noncompact, i.e.,  $\Gamma$  infinite.
- ▶  $\ell^2(\Gamma)$  Hilbert space completion of  $\mathbb{C}[\Gamma]$ .
- ▶ Left action of  $\Gamma$  on  $\ell^2(\Gamma)$ ,  $L(\gamma): \ell^2(\Gamma) \rightarrow \ell^2(\Gamma)$ ,  $\gamma \in \Gamma$ .
- ▶ **Group von Neumann algebra:**  $\mathcal{N}(\Gamma) := \mathcal{B}(\ell^2(\Gamma))^\Gamma$ .
- ▶ **von Neumann trace:**  $\text{tr}_{\mathcal{N}(\Gamma)}: \mathcal{N}(\Gamma) \rightarrow \mathbb{C}$ ,

$$\text{tr}_{\mathcal{N}(\Gamma)}(A) := \langle A(e), e \rangle_{\ell^2(\Gamma)}.$$

- ▶  $\mathcal{B}(L^2\Lambda^p(\tilde{X}))^\Gamma$  von Neumann algebra.
- ▶  $F \subset \tilde{X}$  fundamental domain for the action of  $\Gamma$  on  $\tilde{X}$ .
- ▶  $L^2\Lambda^p(\tilde{X}) \cong \ell^2(\Gamma) \otimes L^2\Lambda^p(F)$ .

- ▶  $\mathcal{B}(L^2\Lambda^p(\tilde{X}))^\Gamma \cong \mathcal{N}(\Gamma) \otimes \mathcal{B}(L^2\Lambda^p(F))$ .
- ▶  $\text{Tr}_\Gamma: \mathcal{N}(\Gamma) \otimes \mathcal{B}(L^2\Lambda^p(F)) \rightarrow \mathbb{C}$  constructed from  $\text{tr}_{\mathcal{N}(\Gamma)}$  on  $\mathcal{N}(\Gamma)$  and the ordinary trace on  $\mathcal{B}(L^2\Lambda^p(F))$ ,  $\text{Tr}_\Gamma$  possibly infinite.
- ▶  $H \subset L^2\Lambda^p(\tilde{X})$   $\Gamma$ -invariant subspace,  $P \in \mathcal{B}(L^2\Lambda^p(\tilde{X}))^\Gamma$  orthogonal projection onto  $H$ . Define

$$\dim_\Gamma H := \text{Tr}_\Gamma(P).$$

### i) $L^2$ -Betti numbers:

- ▶  $\mathcal{H}_{(2)}^p(\tilde{X})$  space of  $L^2$ -harmonic  $p$ -forms of  $\tilde{X}$ .
- ▶  $P: L^2\Lambda^p(\tilde{X}) \rightarrow \mathcal{H}_{(2)}^p(\tilde{X})$  orthogonal projection.
- ▶  $p(x, y)$  kernel of  $P$ ,  $C^\infty$ -kernel.

$$b_p^{(2)}(X) := \dim_\Gamma \mathcal{H}_{(2)}^p(\tilde{X}) = \int_F \text{tr } p(x, x) dx.$$

## ii) $\Gamma$ -index theorem

- ▶  $D: C^\infty(X, E) \rightarrow C^\infty(X, F)$  elliptic differential operator.
- ▶  $\tilde{D}: C^\infty(\tilde{X}, \tilde{E}) \rightarrow C^\infty(\tilde{X}, \tilde{F})$  lift of  $D$  to  $\tilde{X}$ .

$$\text{Ind}_\Gamma(\tilde{D}) := \dim_\Gamma(\ker(\tilde{D}) \cap L^2) - \dim_\Gamma(\ker(\tilde{D}^*) \cap L^2)$$

Theorem (Atiyah):  $\text{Ind}_\Gamma(\tilde{D}) = \text{Ind}(D)$ .

## iii) $L^2$ -torsion

Recall:

$$\log T_X(\rho) = \frac{1}{2} \sum_{p=1}^n (-1)^p p \frac{d}{ds} \zeta_p(s; \rho) \Big|_{s=0}.$$

where

$$\zeta_p(s; \rho) = \frac{1}{\Gamma(s)} \int_0^\infty (\text{Tr}(e^{-t\Delta_p(\rho)}) - b_p(\rho)) t^{s-1} dt, \quad \text{Re}(s) > n/2,$$

and  $b_p(\rho) := \dim \ker \Delta_p(\rho)$ .

Now we consider the corresponding heat operator on  $\tilde{X}$  and replace  $\text{Tr}$  by  $\text{Tr}_\Gamma$ :

- ▶  $\tilde{\Delta}_p(\rho): \Lambda^p(\tilde{X}, \tilde{E}_\rho) \rightarrow \Lambda^p(\tilde{X}, \tilde{E}_\rho)$  lift of  $\Delta_p(\rho)$  to  $\tilde{X}$ .
- ▶  $\tilde{K}_p^\rho(t, x, y)$  kernel of  $e^{-t\tilde{\Delta}_p(\rho)}$

$$\text{Tr}_\Gamma \left( e^{-t\tilde{\Delta}_p(\rho)} \right) = \int_F \text{tr} \tilde{K}_p^\rho(t, x, x) dx.$$

von Neumann trace.

- ▶ We need to determine the asymptotic behavior of  $\text{Tr}_\Gamma \left( e^{-t\tilde{\Delta}_p(\rho)} \right)$  as  $t \rightarrow +0$  and  $t \rightarrow \infty$ .

**a)**  $t \rightarrow +0$ . [Lott, 1992](#): Let  $d$  be the length of the shortest closed geodesic on  $X$ . Put  $N = [n/4] + 1$ . For any  $\varepsilon > 0$  we have

$$\text{Tr}_\Gamma \left( e^{-t\tilde{\Delta}_p(\rho)} \right) - \text{Tr} \left( e^{-t\Delta_p(\rho)} \right) = O(t^{-4N+(1/2)} e^{-(d-\varepsilon)^2/4t}), \quad t \rightarrow 0.$$

**b)**  $t \rightarrow \infty$ . **Novikov-Shubin invariants:**

Let  $b_p^{(2)}(\rho) := \dim_{\Gamma} \ker \tilde{\Delta}_p(\rho)$ . Define  $\tilde{\alpha}_p(X, \rho) \in [0, \infty]$  by

$$\tilde{\alpha}_p(\tilde{X}, \rho) := \sup \left\{ \beta_p : \text{Tr}_{\Gamma}(e^{-t\tilde{\Delta}_p(\rho)}) - b_p^{(2)}(\rho) = O(t^{-\frac{\beta_p}{2}}) \text{ as } t \rightarrow \infty \right\}.$$

**Novikov, Shubin:**  $\rho = 1$ ,  $\tilde{\alpha}_p(\tilde{X})$  are homomorphism invariants.

**Conjecture:**  $\tilde{\alpha}_p(\tilde{X}, \rho) > 0$  for all  $X, \rho$ , and  $p = 0, \dots, n$ .

**Assume:**  $\tilde{\alpha}_p(\tilde{X}, \rho) > 0$ ,  $p = 0, \dots, n$ .

Let  $\tilde{\Delta}_p(\rho)'$  be the restriction of  $\tilde{\Delta}_p(\rho)$  to  $\ker(\tilde{\Delta}_p(\rho))^{\perp}$ . Then  $T_X^{(2)}(\rho) \in \mathbb{R}^+$  is defined by

$$\log T_X^{(2)}(\rho) := \frac{1}{2} \sum_{p=1}^n (-1)^p p \left\{ \frac{d}{ds} \left( \frac{1}{\Gamma(s)} \int_0^1 \text{Tr}_{\Gamma}(e^{-t\tilde{\Delta}_p(\rho)'}) t^{s-1} dt \right) \Big|_{s=0} + \int_1^{\infty} \text{Tr}_{\Gamma}(e^{-t\tilde{\Delta}_p(\rho)'}) t^{-1} dt \right\}.$$

If  $\tilde{\alpha}_p(X, \rho) = \infty$ ,  $p = 0, \dots, n$ , then

$$\log T_X^{(2)}(\rho) := \frac{1}{2} \sum_{p=1}^n (-1)^p p \frac{d}{ds} \left( \frac{1}{\Gamma(s)} \int_0^\infty \text{Tr}_\Gamma(e^{-t\tilde{\Delta}_p(\rho)}) t^{s-1} dt \right) \Big|_{s=0}.$$

### 3. Approximation of $L^2$ -invariants

Assume that  $\Gamma = \pi_1(X)$  is residual finite: There exists a tower

$$\Gamma \supset \Gamma_1 \supset \Gamma_2 \supset \dots \supset \Gamma_N \supset \dots$$

of normal subgroups  $\Gamma_k$  of  $\Gamma$  of finite index with  $\bigcap_{k=1}^\infty \Gamma_k = \{1\}$ .

- ▶ Put  $X_k := \Gamma_k \backslash \tilde{X}$ ,  $k \in \mathbb{N}$ .  $X_k \rightarrow X$  finite normal covering.
- ▶ Let  $\alpha$  be a spectral invariant.

**Problem:** Does  $\frac{\alpha(X_k)}{[\Gamma : \Gamma_k]}$  converge as  $k \rightarrow \infty$  and if so, what is the limit?

- ▶ A natural candidate for the limit is the  $L^2$ -invariant  $\alpha^{(2)}$ .

## Examples:

$\Gamma$ -index theorem:  $D: C^\infty(X, E) \rightarrow C^\infty(X, F)$  elliptic.  $D_k$  lift of  $D$  to  $X_k$ . Atiyah's  $L^2$ -index theorem implies

$$\frac{\text{Ind}(D_k)}{[\Gamma : \Gamma_k]} = \text{Ind}(D) = \text{Ind}_\Gamma(\tilde{D}).$$

Theorem (Lück, 1994):

$$\lim_{k \rightarrow \infty} \frac{b_p(X_k)}{[\Gamma : \Gamma_k]} = b_p^{(2)}(X).$$

► Holds for every finite CW-complex.

**Problem:** Computation of  $b_p^{(2)}(X) = \dim_\Gamma \mathcal{H}_{(2)}^p(\tilde{X})$ . Especially, when is  $b_p^{(2)}(X) \neq 0$ ?

**Conjecture:**  $X$  aspherical closed manifold of dimension  $2n + 1$ .  
Suppose that  $\Gamma = \pi_1(X)$  is residual finite. Then

$$\lim_{j \rightarrow \infty} \frac{\log |H_n(X_j, \mathbb{Z})_{\text{tors}}|}{[\Gamma : \Gamma_j]} = (-1)^n \log T_X^{(2)}(1).$$

**Problem:** Computation of  $T_X^{(2)}(\rho)$ .

### Some applications

**Conjecture (Dodziuk, Singer):** Let  $\dim(X) = 2n$  and  $K < 0$ . Then

$$b_p^{(2)}(X) = 0, \text{ if } p \neq n; \quad b_n^{(2)}(X) > 0.$$

Combined with the  $\Gamma$ -index theorem it implies

**Conjecture (Hopf):** Let  $\dim(X) = 2n$  and  $K < 0$ . Then

$$(-1)^n \chi(X) > 0.$$

### 3. Locally symmetric spaces

- ▶  $G$  semisimple real Lie group of non-compact type
- ▶  $K \subset G$  maximal compact subgroup
- ▶  $\tilde{X} = G/K$  associated Riemannian symmetric space of non-positive curvature
- ▶  $\Gamma \subset G$  a lattice, i.e., discrete subgroup with  $\text{vol}(\Gamma \backslash G) < \infty$
- ▶  $X := \Gamma \backslash \tilde{X}$  locally symmetric space of finite volume.
- ▶  $X$  is an orbifold, a manifold, if  $\Gamma$  is torsion free.

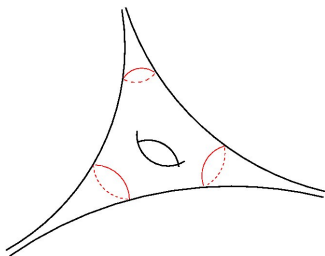
#### Examples:

1.  $G = \text{SL}(2, \mathbb{R})$ ,  $K = \text{SO}(2)$ .  $G/K = \mathbb{H}^2 = \{z \in \mathbb{C} : \text{Im}(z) > 0\}$ .
  - ▶  $\Gamma \subset G$  discrete subgroup, torsion free,  $\Gamma \backslash G$  compact.
  - ▶  $X = \Gamma \backslash \mathbb{H}^2$  is a compact oriented surface.

2. For  $N \in \mathbb{N}$  let

$$\Gamma(N) := \left\{ \gamma \in \mathrm{SL}(2, \mathbb{Z}) : \gamma \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod{N} \right\}$$

- ▶  $\Gamma(N)$  torsion free, if  $N \geq 3$ .
- ▶  $X_N = \Gamma(N) \backslash \mathbb{H}^2$  is a hyperbolic surface of finite area, “surface with cusps”.



A hyperbolic surface with 3 cusps.

Let  $N_1 < N_2 < N_3 < \cdots \rightarrow \infty$  be a sequence in  $\mathbb{N}$  with  $N_k | N_{k+1}$ .  
Let  $\Gamma_k := \Gamma(N_k)$ . Then

$$\Gamma_1 \supset \Gamma_2 \supset \cdots \supset \Gamma_k \supset \cdots$$

is a tower of normal subgroups.

- ▶ Similar construction exists in the cocompact case.

## 4. Computation of $L^2$ -Betti numbers and Novikov-Shubin invariants

- ▶  $\tilde{\Delta}_p: \Lambda^p(\tilde{X}) \rightarrow \Lambda^p(\tilde{X})$  Laplace operator on  $p$ -forms of  $\tilde{X}$ .
- ▶  $\tilde{K}_p(t, x, y)$  integral kernel of  $e^{-t\tilde{\Delta}_p}$ .
- ▶  $F \subset \tilde{X}$  fundamental domain for  $\Gamma$ .

$$b_p^{(2)}(X) = \lim_{t \rightarrow \infty} \text{Tr}_\Gamma \left( e^{-t\tilde{\Delta}_p} \right) = \lim_{t \rightarrow \infty} \int_F \text{tr} \tilde{K}_p(t, x, x) dx.$$

- ▶  $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$  Cartan decomposition,  $T_{x_0} \tilde{X} \cong \mathfrak{p}$ ,  $\text{Ad}_p: K \rightarrow \text{GL}(\mathfrak{p})$  adjoint representation,  $\Lambda^p T^* \tilde{X} \cong G \times_K \Lambda^p \mathfrak{p}^*$  homogeneous vector bundle.

$$L^2 \Lambda^p(\tilde{X}) = L^2(\tilde{X}, \Lambda^p T^* \tilde{X}) \cong [L^2(G) \otimes \Lambda^p \mathfrak{p}^*]^K. \quad (1)$$

- ▶  $e^{-t\tilde{\Delta}_p}$  convolution operator with kernel  $H_t^p: G \rightarrow \text{End}(\Lambda^p \mathfrak{p}^*)$ .

$$e^{-t\tilde{\Delta}_p} \phi(g) = \int_G H_t^p(g^{-1}g')(\phi(g')) dg'.$$

- ▶ Put  $h_t^p(g) = \text{tr } H_t(g)$ ,  $g \in G$ . Then  $\text{tr } \tilde{K}_p(t, x, x) = h_t^p(e)$ , and  $\text{Tr}_\Gamma \left( e^{-t\tilde{\Delta}_p} \right) = \text{vol}(X)h_t^p(e)$ . Hence

$$b_p^{(2)}(X) = \text{vol}(X) \lim_{t \rightarrow \infty} h_t(e).$$

- ▶ Use the Plancherel theorem (spectral resolution) to study  $h_t(e)$  as  $t \rightarrow \infty$ .
- ▶  $R: G \rightarrow \text{Aut}(L^2(G))$  right regular representation of  $G$ :

$$(R(g)f)(g') := f(g'g), \quad g, g' \in G.$$

- ▶  $\Omega \in \mathcal{Z}(\mathfrak{g}_\mathbb{C})$  Casimir element.

**Lemma (Kuga):**  $\tilde{\Delta}_p = -R(\Omega)$ , with respect to the isomorphism (1).

- ▶ Spectral decomposition of  $\tilde{\Delta}_p$  can be deduced from the spectral decomposition of  $(R, L^2(G))$ .

## 5. Harish-Chandra's Plancherel theorem

- ▶  $P \subset G$  parabolic subgroup,  $P = MAN$ ,  $M$  reductive,  $A \cong (\mathbb{R}^+)^k$ ,  $N$  nilpotent.  $P$  cuspidal parabolic, if  $\widehat{M}_d \neq \emptyset$  (discrete series, matrix coefficients are in  $L^2$ ).
- ▶  $\mathfrak{a} = \text{Lie}(A)$ ,  $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ ,  $\nu \in \widehat{M}_d$ ,

$$H_{\nu, \lambda} = \left\{ f: G \rightarrow W_{\nu} \mid f(gman) = e^{-(\lambda + \rho_{\mathfrak{a}})(\log a)} \nu(m)^{-1} f(g), \right. \\ \left. \forall g \in G, \forall man \in MAN, f|_K \in L^2(K, W_{\nu}) \right\} \\ (\pi_{\nu, \lambda}(g)f)(x) := f(g^{-1}x). \tag{2}$$

- ▶  $(\pi_{\nu, \lambda}, H_{\nu, \lambda})$  induced representation, if  $\lambda \in i\mathfrak{a}^*$ , then  $\pi_{\nu, \lambda}$  is unitary.
- ▶  $\Omega_M \in \mathcal{Z}(\mathfrak{m}_{\mathbb{C}})$  Casimir element.

$$\pi_{\nu, \lambda}(\Omega) = \langle \nu, \nu \rangle - \langle \rho_{\mathfrak{a}}, \rho_{\mathfrak{a}} \rangle + \nu(\Omega_M).$$

- ▶ Let  $\nu \in i\mathfrak{a}^*$ ,  $f \in \mathcal{C}(G)$ ,

$$\pi_{\nu,\lambda}(f) := \int_G f(g)\pi_{\nu,\lambda}(g)dg.$$

- ▶  $\pi_{\nu,\lambda}(f)$  trace class operator in  $H_{\nu,\lambda}$ .
- ▶  $p_{\nu}(i\lambda)$  Plancherel density.

**Theorem (Harish-Chandra):** For all  $f \in \mathcal{C}(G)_{K \times K}$  one has

$$f(g) = \sum_P \sum_{\nu \in \widehat{M}_d} \int_{\mathfrak{a}^*} \text{Tr}(\pi_{\nu,i\lambda}(f)\pi_{\nu,i\lambda}(g^{-1}))p_{\nu}(i\lambda)d\lambda.$$

- ▶ Let  $K_M := K \cap M$ , maximal compact subgroup of  $M$ .
- ▶  $h_t \in \mathcal{C}(G)_{K \times K}$ .

Application of Harish-Chandra's theorem gives

$$h_t(e) =$$

$$\sum_P \sum_{\nu \in \widehat{M}_d} \int_{\mathfrak{a}^*} e^{-t(\|\lambda\|^2 + \|\rho_{\mathfrak{a}}\|^2 - \nu(\Omega_M))} \dim[H_{\nu} \otimes \Lambda^p \mathfrak{p}^*]^{K_M} p_{\nu}(i\lambda)d\lambda.$$

- ▶  $\delta(\tilde{X}) := \text{rank}(\mathfrak{g}_{\mathbb{C}}) - \text{rank}(\mathfrak{k}_{\mathbb{C}})$  fundamental rank of  $\tilde{X}$ .

### Examples:

- ▶  $\delta(\tilde{X}) = 0$  for  $SU_{p,q}$ ,  $SO_{p,q}$  ( $pq$  even),  $\mathbb{H}_{\mathbb{C}}^n = SU(n, 1)/SU(n)$  complex hyperbolic space.
- ▶  $\delta(\tilde{X}) = 1$  for  $SO_{p,q}$  ( $pq$  odd),  $SL_3(\mathbb{R})$ ,  $SL_4(\mathbb{R})$ ,  $\mathbb{H}^n = SO^0(n, 1)/SO(n)$  hyperbolic space,  $n$  odd.
- ▶  $\delta(\tilde{X}) \geq 2$  for  $SL_n(\mathbb{R})$ ,  $n \geq 5$ .

**Theorem (Lohoue, Olbrich, ...)** Let  $n = \dim X$ .

- 1)  $b_p^{(2)}(X) \neq 0 \Leftrightarrow \delta(\tilde{X}) = 0$  and  $p = n/2$ . In particular,  $b_{\frac{n}{2}}^{(2)}(X) = (-1)^{\frac{n}{2}} \chi(X)$ .
- 2)  $\alpha_p(\tilde{X}) \neq +\infty \Leftrightarrow \delta := \delta(\tilde{X}) > 0$  and  $p \in [(n - \delta)/2, (n + \delta)/2]$ . In this case  $\alpha_p(\tilde{X}) = \delta$ .
- 3)  $T_X^{(2)}(1) \neq 1 \Leftrightarrow \delta(\tilde{X}) = 1$ .

- ▶ Lück's theorem gives no information about cohomology of hyperbolic 3-manifolds.

### Extension to $\rho \neq 1$

- ▶  $\rho: G \rightarrow \mathrm{GL}(V)$  finite dimensional complex representation.
- ▶  $F_\rho \rightarrow \Gamma \backslash \tilde{X}$  flat vector bundle associated to  $\rho|_\Gamma$ .
- ▶  $\tilde{E}_\rho = G \times_K V \rightarrow \tilde{X}$  homogeneous vector bundle associated to  $\rho|_K$ .

**Lemma (Matsushima, Murakami):** There is a canonical isomorphism  $F_\rho \cong \Gamma \backslash \tilde{E}_\rho$ .

- ▶  $\tilde{E}_\rho$  equipped with  $G$ -invariant metric, pushed down to  $F_\rho \cong \Gamma \backslash \tilde{E}_\rho$ .

**Theorem (Mü, Pfaff, 2014):** Let  $\rho: G \rightarrow \mathrm{GL}(V)$  be irreducible. Let  $\theta: G \rightarrow G$  be the Cartan involution. Assume that  $\rho \circ \theta \not\cong \rho$ . Then there exists  $c > 0$  such that

$$\mathrm{Tr}_\Gamma \left( e^{-t\tilde{\Delta}_\rho(\rho)} \right) = O(e^{-ct}), \quad t \rightarrow \infty.$$

## 5. Analytic torsion and $L^2$ -torsion

Basic setup:

- ▶  $\tilde{X} := G/K$ ,  $\Gamma \subset G$  cocompact lattice,  $X = \Gamma \backslash \tilde{X}$ .
- ▶  $\Gamma_k \subset \Gamma$ ,  $k \in \mathbb{N}$ , a sequence of congruence subgroups.
- ▶  $X_k := \Gamma_k \backslash \tilde{X}$ ,  $k \in \mathbb{N}$ , finite covering of  $X$ .
- ▶  $\rho: G \rightarrow \mathrm{GL}(V)$  a finite-dimensional representation,  
 $\rho_k := \rho|_{\Gamma_k}: \Gamma_k \rightarrow \mathrm{GL}(V)$ .
- ▶  $E_k \rightarrow X_k$  flat bundle associated to  $\rho_k$ ,
- ▶  $\Delta_{X_k, \rho}(\rho)$  the Laplacian on  $E_k$ -valued  $p$ -forms on  $X_k$ .

$\rho$  is called **strongly acyclic**, if there exists  $c > 0$  such that

$$\mathrm{Spec}(\Delta_{X_k, \rho}(\rho)) \subset [c, \infty)$$

for all  $k \in \mathbb{N}$  and  $p = 0, \dots, n$ .

**Lemma:**  $\theta: G \rightarrow G$  Cartan involution. Assume that  $\rho$  is irreducible and  $\rho \circ \theta \not\cong \rho$ . Then  $\rho$  is strongly acyclic.

**Example:**  $G = \mathrm{SL}(2, \mathbb{C})$ ,  $\mathbb{H}^3 = \mathrm{SL}(2, \mathbb{C}) / \mathrm{SU}(2)$  hyperbolic 3-space. The real representation

$$\rho_{p,q} := \mathrm{Sym}^p(\mathbb{C}^2) \otimes \overline{\mathrm{Sym}^q(\mathbb{C}^2)}$$

of  $\mathrm{SL}_2(\mathbb{C})$  is strongly acyclic if and only if  $p \neq q$ .

► If  $\rho$  is strongly acyclic, then

$$\mathrm{Tr}_\Gamma(e^{-t\tilde{\Delta}_\rho(\rho)}) = O(e^{-ct}), \quad t \rightarrow \infty.$$

Hence the  $L^2$ -torsion  $T_X^{(2)}(\rho)$  is well defined.

**Theorem (Bergeron, Venkatesh, 2009):** Let  $\Gamma \subset G$  be cocompact. Let  $\rho: G \rightarrow \mathrm{GL}(V)$  be strongly acyclic. Let  $\Gamma_k$  be sequence of congruence subgroups of  $\Gamma$  for which the injectivity radius of  $X_k = \Gamma_k \backslash \tilde{X}$  goes to infinity. Then

$$\lim_{k \rightarrow \infty} \frac{\log T_{X_k}(\rho)}{[\Gamma : \Gamma_k]} = \log T_X^{(2)}(\rho).$$

### Sketch of the proof

We have

$$\log T_X(\rho) = \frac{1}{2} \sum_{p=1}^n (-1)^p p \frac{d}{ds} \zeta_p(s, \rho) \Big|_{s=0}$$

and

$$\zeta_p(s, \rho) = \frac{1}{\Gamma(s)} \int_0^\infty \mathrm{Tr} \left( e^{-t\Delta_\rho(\rho)} \right) t^{s-1} dt$$

Let  $A > 0$ . Then

$$\begin{aligned} \frac{d}{ds} \zeta_P(s, \rho) \Big|_{s=0} &= \frac{d}{ds} \left( \frac{1}{\Gamma(s)} \int_0^A \text{Tr} \left( e^{-t\Delta_p(\rho)} \right) t^{s-1} dt \right) \Big|_{s=0} \\ &\quad + \int_A^\infty t^{-1} \text{Tr} \left( e^{-t\Delta_p(\rho)} \right) dt. \end{aligned}$$

If  $\rho$  is strongly acyclic, then for all  $\varepsilon > 0$  there exists  $A > 0$  s.th.

$$\frac{1}{[\Gamma : \Gamma_k]} \int_A^\infty t^{-1} \text{Tr} \left( e^{-t\Delta_{X_k, \rho}(\rho)} \right) dt \leq \varepsilon$$

for all  $k \in \mathbb{N}$ .

To deal with the first term, we apply the [Selberg trace formula](#).

- ▶ Let  $p = 0$ ,  $\rho = 1$ ,  $X = \Gamma \backslash \tilde{X}$ .
- ▶  $\Delta: C^\infty(X) \rightarrow C^\infty(X)$  Laplace operator.
- ▶  $K(t, x, y)$  kernel of  $e^{-t\Delta}$ .

- ▶  $\tilde{\Delta}$  Laplace operator of  $\tilde{X}$ ,  $k(t, x, y)$  kernel of  $e^{-t\tilde{\Delta}}$ .
- ▶ There exists  $h_t \in C^\infty(\mathbb{R})$  such that  $k(t, x, y) = h_t(d(x, y))$ .

$$K(t, x, y) = \sum_{\gamma \in \Gamma} k(t, x, \gamma(y)).$$

$$\text{Tr}(e^{-t\Delta}) = \int_X K(t, x, x) dx = \int_X \sum_{\gamma \in \Gamma} k(t, x, \gamma(x)) dx.$$

- ▶  $\ell(X)$  length of shortest closed geodesic of  $X$ .

Using standard heat kernel estimations, it follows that there exist  $C, c > 0$ , depending only on  $(G, A)$  such that

$$\frac{1}{\text{vol}(X)} \left| \int_X \sum_{\gamma \in \Gamma - \{1\}} k(t, x, \gamma(x)) dx \right| \leq Ct^{-(n+1)} e^{-(\ell(X) - ct)^2 / 5t}$$

for  $0 < t \leq A$ .

Moreover

$$\int_X k(t, x, x) dx = h_t(0) \operatorname{vol}(X).$$

This implies

$$\frac{\log T_{X_k}(\rho)}{[\Gamma : \Gamma_k]} \rightarrow t_{\tilde{X}}^{(2)}(\rho) \operatorname{vol}(X), \quad \ell(X_k) \rightarrow \infty.$$

- ▶  $t_{\tilde{X}}^{(2)}(\rho)$  depends only on  $\tilde{X}$  and  $\rho$  and can be computed using the Plancherel formula.

**Theorem:** If  $\delta(G) = \operatorname{rank}_{\mathbb{C}}(G) - \operatorname{rank}_{\mathbb{C}}(K) = 1$ , then  $t_{\tilde{X}}^{(2)}(\rho) \neq 0$ .

- ▶ If  $\rho$  is not strongly acyclic, eigenvalues of  $\Delta_{X_k, \rho}(\rho)$  may cluster near zero as  $k \rightarrow \infty$ . Then the method does not work.

**Problem:** Study behavior of  $\log T_{X_k}(\rho)/[\Gamma : \Gamma_k]$  as  $k \rightarrow \infty$  without restrictions on  $\rho$ . Especially for  $\rho = 1$ .

- ▶  $X = \Gamma \backslash \mathbb{H}^n$ ,  $\Gamma \subset \mathrm{SO}_0(n, 1)$ , compact hyperbolic  $n$ -manifold.
- ▶  $\lambda_1^k(X) > 0$  first positive eigenvalue of  $\Delta_k$  on  $X$ .

**Theorem (Bergeron, Clozel, 2013):** For all  $0 \leq k \leq n/2 - 1$  there exists  $c(n, k) > 0$  such that  $\lambda_1^k(X) \geq c(n, k)$  for all congruence subgroups of  $\Gamma \subset \mathrm{SO}_0(n, 1)$ .

- ▶  $(\Gamma_i)_{i \in \mathbb{N}}$  sequence of cocompact congruence subgroups of  $\Gamma$ .
- ▶  $n = 2d + 1$ ,  $\lambda_j^{(i)}$  eigenvalues of  $\Delta_d$  on  $\Lambda^d(X_i)$ .

**Theorem:** Assume

- 1) For every  $\varepsilon > 0$  there exists  $c > 0$  such that

$$\limsup_{i \rightarrow \infty} \frac{1}{\mathrm{vol}(X_i)} \sum_{0 < \lambda_j^{(i)} \leq c} |\log \lambda_j^{(i)}| \leq \varepsilon.$$

$$2) \dim H_d(X_i; \mathbb{Q}) = o\left(\frac{\text{vol}(X_i)}{\log \text{vol}(X_i)}\right), i \rightarrow \infty.$$

Then

$$\frac{\log T_{X_i}(1)}{\text{vol}(X_i)} \rightarrow (-1)^d t_{\mathbb{H}^n}^{(2)}.$$

**Theorem (R. Schoen, 1982:)**  $\lambda_1(X) > 0$  first positive eigenvalue of  $\Delta_0$  on compact hyperbolic manifolds  $X$  of dimension  $n$ . For  $n \geq 3$  there exists  $c(n) > 0$  such that

$$\lambda_1(X) \geq \frac{c(n)}{\text{vol}(X)^2}$$

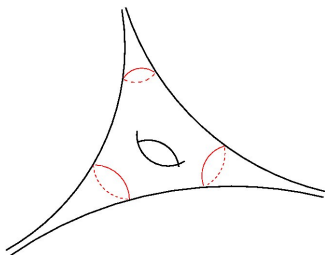
for all  $X$  of dimension  $n$ .

- ▶ There are no such estimates known for the first positive eigenvalue of  $\Delta_p$  for  $p \geq 1$ .

## 5. The finite volume case

Standard arithmetic groups like  $SL(2, \mathbb{Z}[i]) \subset SL(2, \mathbb{C})$  or  $SL(n, \mathbb{Z}) \subset SL(n, \mathbb{R})$  are not cocompact. Extension of the results to these groups is very desirable.

- ▶ If  $X = \Gamma \backslash \tilde{X}$  is not compact, but has finite volume, then the Laplace operators have non-empty continuous spectrum.
- ▶ The zeta function can not be defined in the usual way.
- ▶ Regularization of the trace of the heat operator is necessary.
- ▶  $X = \Gamma \backslash \mathbb{H}^n$  hyperbolic manifold of finite volume.  $X$  is manifold with finitely many cusp ends.



- ▶ Use  $b$ -trace of Melrose.
- ▶  $Y > 0$  sufficiently large.  $X_Y$  truncation of  $X$  at level  $Y$ .

**Proposition.** Let  $K_\rho^\rho(t, x, y)$  be the kernel of  $e^{-t\Delta_\rho(\rho)}$ . One has

$$\int_{X_Y} \text{tr} K_\rho^\rho(t, x, x) dx = a(t) \log Y + b(t) + O(Y^{-1}), \quad Y \rightarrow \infty.$$

Put

$$\text{Tr}_{\text{reg}} \left( e^{-t\Delta_\rho(\rho)} \right) := b(t)$$

- ▶  $\text{Tr}_{\text{reg}} \left( e^{-t\Delta_\rho(\rho)} \right)$  equals the spectral side of the Selberg trace formula.

**Example:**

- ▶  $X = \Gamma \backslash \mathbb{H}^2$ ,  $\Gamma \subset \text{SL}(2, \mathbb{R})$  lattice,  $X$  hyperbolic surface.
- ▶  $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \dots$  eigenvalues of  $\Delta_0$ .
- ▶  $C(s) = (C_{kl}(s))$  scattering matrix,  $\phi(s) = \det C(s)$ .

$$\text{Tr}_{\text{reg}} \left( e^{-t\Delta_0} \right) = \sum_j e^{-t\lambda_j} - \frac{1}{4\pi} \int_{\mathbb{R}} e^{-(1/4+\lambda^2)t} t \frac{\phi'}{\phi} (1/2 + i\lambda) d\lambda + \dots$$

**Proposition:** Let  $n = \dim(X)$  be odd,  $\rho \circ \theta \not\cong \rho$ . As  $t \rightarrow +0$

$$\mathrm{Tr}_{\mathrm{reg}} \left( e^{-t\Delta_{\rho}(\rho)} \right) \sim \sum_{j=0}^{\infty} t^{j-n/2} + \sum_{j=0}^{\infty} b_j t^{j-\frac{1}{2}} \log t + \sum_{j=0}^{\infty} c_j t^j.$$

As  $t \rightarrow \infty$ ,

$$\mathrm{Tr}_{\mathrm{reg}} \left( e^{-t\Delta_{\rho}(\rho)} \right) = O(e^{-ct}).$$

**Method:** Albin, Rochon:  $b$ -calculus of Melrose; Mü, Pfaff: Selberg trace formula.

Let

$$\zeta_{\rho}^{\mathrm{reg}}(s, \rho) := \frac{1}{\Gamma(s)} \int_0^{\infty} \mathrm{Tr}_{\mathrm{reg}} \left( e^{-t\Delta_{\rho}(\rho)} \right) t^{s-1} dt, \quad \mathrm{Re}(s) > n/2.$$

- ▶  $\zeta_{\rho}^{\mathrm{reg}}(s, \rho)$  admits meromorphic extension to  $\mathbb{C}$ , holomorphic at  $s = 0$ .

- ▶ The regularized analytic torsion  $T_X^{\text{reg}}(\rho) \in \mathbb{C}^\times$  can be defined as in the compact case by

$$\log T_X^{\text{reg}}(\rho) := \frac{1}{2} \sum_{p=1}^n (-1)^p p \frac{d}{ds} \zeta_p^{\text{reg}}(s; \rho) \Big|_{s=0}.$$

**Theorem (Mü., Pfaff, 2014):** Let  $\Gamma \subset \text{SO}_0(n, 1)$  be a lattice and  $\Gamma_k \subset \Gamma$  a sequence of finite index normal subgroups, which are torsion free, cusp uniform, and satisfy  $[\Gamma : \Gamma_k] \rightarrow \infty$  as  $k \rightarrow \infty$ . Let  $X_k := \Gamma_k \backslash \mathbb{H}^n$  and let  $\rho: \text{SO}_0(n, 1) \rightarrow \text{GL}(V)$  be strongly acyclic. Then

$$\lim_{k \rightarrow \infty} \frac{\log T_{X_k}^{\text{reg}}(\rho)}{[\Gamma : \Gamma_k]} = t_{\tilde{X}}^{(2)}(\rho) \text{vol}(X).$$

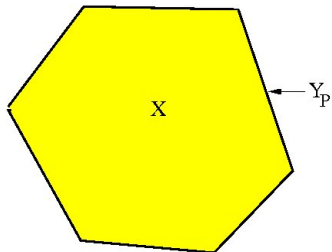
- ▶ The case of hyperbolic 3-manifolds was treated by J. Raimbault.

2. Higher rank Joint work with Jasmin Matz.

Regularized trace of heat operator

Use **Borel-Serre compactification**. Manifold with corners.

$$\overline{X}^{BS} = X \sqcup_{\{P\}} Y_P.$$



- ▶ Pass to adelic framework.
- ▶  $\mathbf{G}$  semisimple algebraic group over  $\mathbb{Q}$ ,  
 $G(\mathbb{A}) = \prod'_v G(\mathbb{Q}_v) = G(\mathbb{R}) \times \prod'_p G(\mathbb{Q}_p)$
- ▶  $K_f \subset \mathbf{G}(\mathbb{A}_f)$  open compact subgroup,  $\tilde{X} = \mathbf{G}(\mathbb{R})/K_\infty$ ,
- ▶ adelic locally symmetric space  
 $X(K_f) = \mathbf{G}(\mathbb{Q}) \backslash (\tilde{X} \times G(\mathbb{A}_f)) / K_f$ .
- ▶ There exist lattices  $\Gamma_i \subset \mathbf{G}(\mathbb{R})$ ,  $i = 1, \dots, m$ , such that

$$X(K_f) = \Gamma_1 \backslash \tilde{X} \sqcup \dots \sqcup \Gamma_m \backslash \tilde{X}.$$

- ▶  $\mathbf{G}$  simply connected, then by strong approximation

$$X(K_f) = \Gamma \backslash \tilde{X}, \quad \Gamma = (\mathbf{G}(\mathbb{R}) \times K_f) \cap \mathbf{G}(\mathbb{Q}).$$

- ▶ We use Arthur's trace formula to define the regularized trace of the heat operators.

- ▶  $h_t$  kernel of  $e^{-t\tilde{\Delta}}$  on  $\tilde{X} = \mathbf{G}(\mathbb{R})/K_\infty$ ,  $\phi_t := h_t \otimes \mathbf{1}_{K_f}$ ,  $\phi_t \in \mathcal{C}(\mathbf{G}(\mathbb{A}); K_f)$ .
- ▶ Let  $J_{\text{spec}}(\phi_t)$  be the spectral side of the Arthur trace formula.

We define the regularized trace of the heat operator by

$$\text{Tr}_{\text{reg}}(e^{-t\Delta}) := J_{\text{spec}}(\phi_t).$$

If  $X(K_f(N)) = \Gamma(N)\backslash\mathbb{H}^2$ , then

$$J_{\text{spec}}(\phi_t) = \sum_j e^{-t\lambda_j} - \frac{1}{4\pi} \int_R e^{-(1/4+r^2)t} \frac{\phi'}{\phi}(1/2 + ir) dr + \dots,$$

where  $\lambda_j$  are the eigenvalues of  $\Delta$ ,  $\phi(s) = \det C(s)$ , and  $C(s) = (C_{ij}(s))$  is the scattering matrix.

The regularized analytic torsion is then defined by

$$\log T_{X(K_f)}^{\text{reg}}(\rho) = \frac{1}{2} \sum_{q=1}^n (-1)^q q \cdot \left. \frac{d}{ds} \left( \frac{1}{\Gamma(s)} \int_0^\infty \text{Tr}_{\text{reg}}(e^{-t\Delta_q(\rho)}) t^{s-1} dt \right) \right|_{s=0}.$$

Let  $\tilde{X}_n = \text{SL}(n, \mathbb{R}) / \text{SO}(n)$ ,  $\Gamma(N) \subset \text{SL}(n, \mathbb{Z})$  principal congruence subgroup,  $X_n(N) = \Gamma(N) \backslash \tilde{X}_n$ ,  $\tau \in \text{Rep}(\text{SL}(n, \mathbb{R}))$ ,  $\tau_\theta = \tau \circ \theta$ , where  $\theta$  is the Cartan involution.

**Theorem (Matz, M.):** Let  $\tau \in \text{Rep}(\text{SL}(n, \mathbb{R}))$ . Assume  $\tau \not\cong \tau_\theta$ . Then for  $n \geq 2$  we have

$$\lim_{N \rightarrow \infty} \frac{\log T_{X_n(N)}^{\text{reg}}(\tau)}{\text{vol}(X_n(N))} = t_{\tilde{X}_n}^{(2)}(\tau).$$

Moreover, if  $n > 4$ , then  $t_{\tilde{X}_n}^{(2)}(\tau) = 0$ , and if  $n = 3, 4$ , then  $t_{\tilde{X}_n}^{(2)}(\tau) > 0$ .

## Applications

- ▶  $M \subset V_\rho$   $\pi_1(X)$ -invariant lattice,  $\mathcal{M}$  associated local system of free  $\mathbb{Z}$ -modules over  $X$ .
- ▶ Hodge isomorphism  $H^p(X; \mathcal{M} \otimes \mathbb{R}) \cong \mathcal{H}^p(X; \mathcal{M} \otimes \mathbb{R})$ , introduces inner product in  $H^p(X; \mathcal{M} \otimes \mathbb{R})$ .

$$R_p(\mathcal{M}) = \text{vol}(H^p(X; \mathcal{M} \otimes \mathbb{R})/H^p(X; \mathcal{M})_{\text{free}})$$

The regulator  $R(\mathcal{M})$  is defined as

$$R(\mathcal{M}) := \prod_{p=0}^n R_p(\mathcal{M})^{(-1)^p}.$$

Theorem (Cheeger, Mü)

$$T_X(\rho) = R(\mathcal{M}) \cdot \prod_{p=0}^n \left| H^p(X; \mathcal{M})_{\text{tors}} \right|^{(-1)^{p+1}}.$$

## Theorem (Bergeron-Venkatesh):

Let  $M$  be strongly acyclic. Then

$$\lim_{N \rightarrow \infty} \sum_{p=0}^n (-1)^{p+1} \frac{\log |H^p(\Gamma_N; M)|}{[\Gamma : \Gamma_N]} = \text{vol}(X) t_{\tilde{X}}^{(2)}(M).$$

If  $\delta(G) = 1$ , we have  $t_{\tilde{X}}^{(2)}(M) \neq 0$ . Then  $\dim \tilde{X}$  is odd. It follows that

$$\liminf_N \sum_p \frac{\log |H^p(\Gamma_N; M)|}{[\Gamma : \Gamma_N]} \geq C_{G,M} \text{vol}(X),$$

where  $p$  is taken over integers with the same parity as  $\frac{\dim X - 1}{2}$  and  $C_{G,M} > 0$ .

## Finite volume case

### Example:

Let  $a_1, \dots, a_d \in \mathbb{N}$  and let  $q$  be the quadratic form defined by

$$q(x_1, \dots, x_d, x_{d+1}) = a_1 x_1^2 + \dots + a_d x_d^2 - x_{d+1}^2.$$

- ▶ Let  $\mathbf{G}$  be the algebraic group defined by  $q$ . Then  $\mathbf{G}$  is defined over  $\mathbb{Q}$  and  $G := \mathbf{G}(\mathbb{R}) \cong \mathrm{SO}(d, 1)$ .
- ▶  $\Gamma := \mathbf{G}(\mathbb{Z})$  is a lattice in  $G$ . For  $d \geq 4$ ,  $\mathbf{G}(\mathbb{Z})$  is not co-compact.  $X = \Gamma \backslash \mathbb{H}^d$ .
- ▶  $\rho: \mathbf{G} \rightarrow \mathrm{GL}(V)$  rational representation,  $\Lambda \subset V$   $\Gamma$ -invariant lattice.  $\Gamma(N) \subset \Gamma$  principal congruence subgroup,  $X_j := \Gamma(j) \backslash \mathbb{H}^d$ .

**Theorem (M., Rochon):** Let  $\Lambda$  be strongly  $L^2$ -acyclic. Let  $E_j = \mathbb{H}^d \times_{\rho|_{\Gamma(j)}} \Lambda$  be the bundle of free  $\mathbb{Z}$ -modules over  $X_j$ . Assume that  $E_j \cong E_j^*$ . Let  $d = 2n + 1$ . Then we have

$$\liminf_{j \rightarrow \infty} \frac{\sum_{q \text{ even}} \log |H^q(X_j; E_j)_{\text{tors}}|}{[\Gamma : \Gamma(j)]} \geq (-1)^n t_{\mathbb{H}^d}^{(2)}(\rho) \text{vol}(\Gamma \backslash \mathbb{H}^d) > 0.$$