Extra twisted connected sums and their ν-invariants

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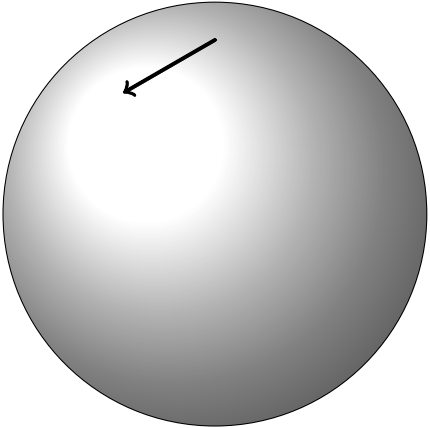
Geometry, Dynamics, and Zeta Functions
Nijmegen, 17–19. 6. 2026

Outline

- G_2-Geometry
  Intro, properties, questions
- The ν-invariant
  Differential topology, definition of ν, properties, first examples
- Extra twisted connected sums
  Construction, properties, problems
- Computation of the ν-invariant
  Computations with η-invariants, examples, questions

Consider parallel translation along a spherical triangle



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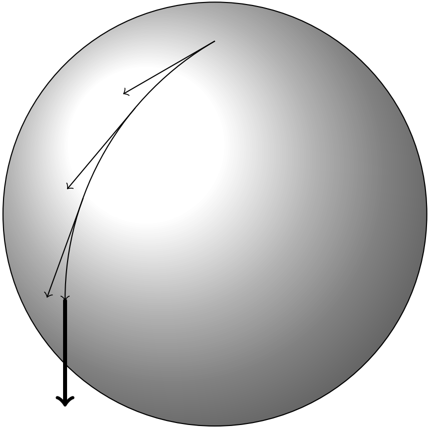

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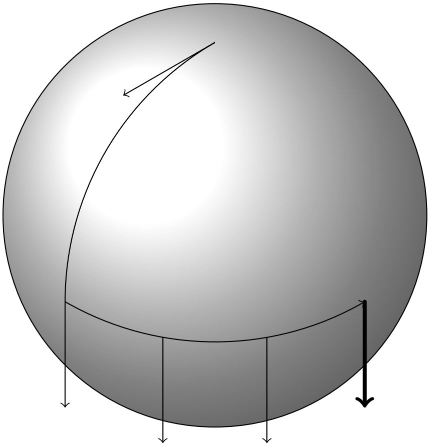

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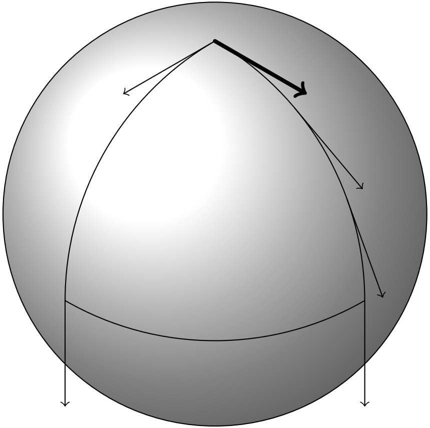

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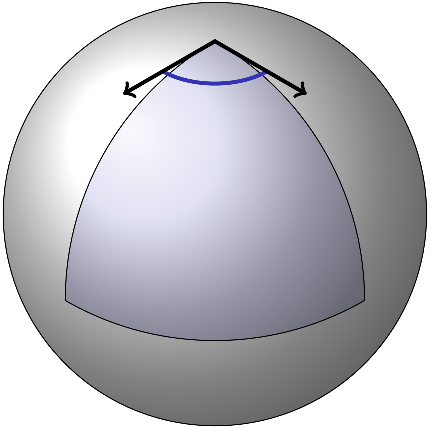

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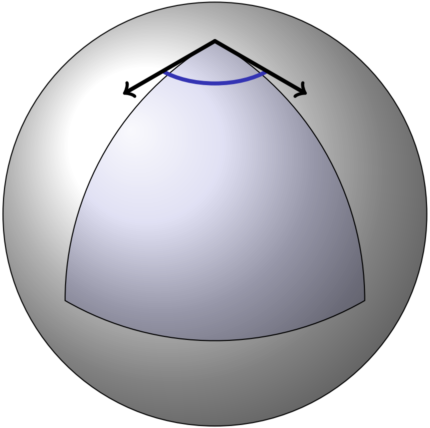
$$SO(2) \cong \mathrm{Hol}(S^2, g^{\mathrm{rnd}}) \subset \mathrm{Aut}(T_pS^2)$$



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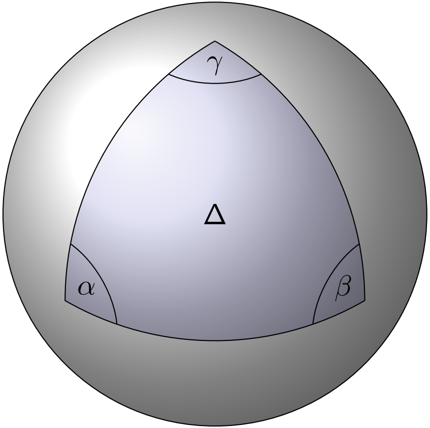
Related: the spherical area formula

$$A(\Delta) = \alpha + \beta + \gamma - \pi \ .$$



Theorem (Berger)

The only possible holonomy groups of complete, simply connected Riemannian manifolds that are neither a product nor a symmetric space are

Holonomy group	dim	ric	*Structure*	*Parallel spinors*	*Name*
$SO(n)$	n				*generic case*
$U(k)$	$2k$		J		*Kähler*
$SU(k)$	$2k$	0	J, Ω	2	*Calabi-Yau*
$Sp(\ell) \cdot Sp(1)$	4ℓ	const	$\langle I, J, K \rangle$		*Quat. Kähler*
$Sp(\ell)$	4ℓ	0	I, J, K, Ω	$\ell + 1$	*hyper Kähler*
G_2	7	0	$\varphi \in \Omega^3$	1	*exceptional*
Spin(7)	8	0	$\psi \in \Omega^4$	1	*exceptional*

Mathematical motivation

- Only special holonomy group for odd dimensional manifolds
- Only G_2 and $\mathrm{Spin}(7)$ holonomy have no direct relation to algebraic geometry

Hence, new methods are needed

G_2-Geometry—Some motivation

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Physical motivation

- In string theory, spacetime takes the form $\mathbb{R}^{3,1} \times V$, where V is Calabi-Yau
- In M-theory, spacetime takes the form $\mathbb{R}^{3,1} \times M$, where M is a G_2-manifold
- Possible relations to other physical theories

Hence, many fruitful interactions possible

G_2 is simply connected $\rightsquigarrow$ G_2-manifolds are spin
The stabiliser of a nonzero spinor in $\mathrm{Spin}(7)$ is G_2
A Riemannian 7-manifold (M, g) has $\mathrm{Hol}(M, g) \subset G_2$
if and only if it is spin and there exists a nonzero parallel spinor

Let M be a simply connected compact oriented spin 7-manifold and define

$$\mathcal{X} = \{\, g \mid \mathrm{Hol}(M, g) \cong G_2 \,\}$$

Let $\mathcal{D} \subset \mathrm{Diff}(M)$ be the connected component of id_M.

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The space $\mathcal{M} = \mathcal{X}/\mathcal{D}$ is a manifold of dimension $b_3(M)$.

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It can be disconnected [Crowley-G-Nordström]
and contain nontrivial 2-spheres [G-Hertl]

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- Construct G_2-metrics with prescribed singularities
  Certain singularities allow massless chiral Fermions to appear in M-theory

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Idea. Use nowhere vanishing spinors to describe and distinguish G_2-structures

Let σ_0, σ_1 be two nowhere vanishing spinors. Extend to $\bar{\sigma} \in \Gamma(\mathcal{S}^+(M \times [0,1]))$
A generic $\bar{\sigma}$ will have nondegenerate isolated zeros because

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Orient $\mathcal{S}^+(M \times [0,1])$ and count with signs

$$\Delta\nu(M; \sigma_0, \sigma_1) = 2 \cdot \#\bar{\sigma}^{-1}(0) = 2 \cdot \sum_{p \in \bar{\sigma}^{-1}(0)} \operatorname{sign}(d_p \bar{\sigma})$$

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Can we write $\Delta\nu(M; \sigma_0, \sigma_1) = \nu(M, \sigma_0) - \nu(M, \sigma_1) \in \mathbb{Z}/48$?

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Extend σ to $\bar{\sigma} \in \Gamma(\mathcal{S}^+ W)$, then $\#\bar{\sigma}^{-1}(0)$ depends on *W*—not well-defined yet!

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Assume that $M = \partial W$ with W spin, compact. Define

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Problem. Given M, how to determine W with $M = \partial W$?

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- $\psi(\nabla^{\mathcal{S}M}, g^{SM})$—Mathai-Quillen form in $\Omega^\bullet(\mathcal{S}M)$
- D_M—spin Dirac operator on $\Gamma(\mathcal{S}M)$
- B_M—odd signature operator $*d \pm d*$ on $\Omega^{\mathrm{ev}}(M)$
- h—dimension of the kernel
- η—Atiyah-Patodi-Singer η-invariant

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Theorem (Crowley-G-Nordström)

$$\nu(M,\sigma) = 2\int_M \sigma^*\psi(\nabla^{SM}, g^{SM}) - 24(\eta + h)(D_M) + 3\eta(B_M) \in \mathbb{Z}/48$$

Proof.

Use $2e(\nabla^{\mathcal{S}^+W}) = e(\nabla) + 48\hat{A}(\nabla)^{[8]} - 3L(\nabla)^{[8]} \in \Omega^8(W)$ □

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- σ is parallel, so $\sigma^*\psi(\nabla^{SM}, g^{SM}) = 0$

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Let (M, g) be a compact manifold with $\mathrm{Hol}(M, g) \subset G_2$. Put

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- $\bar{\nu}(M, g) = 0$ if M admits an orientation reversing isometry

The ν-invariant—First examples and questions

What about the known examples by Joyce and Kovalev?

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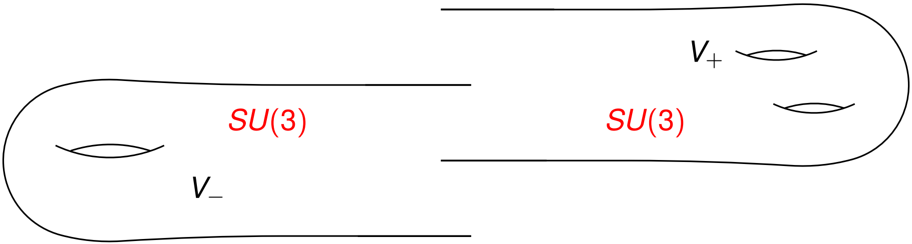
—even with the same underlying G_2-structure

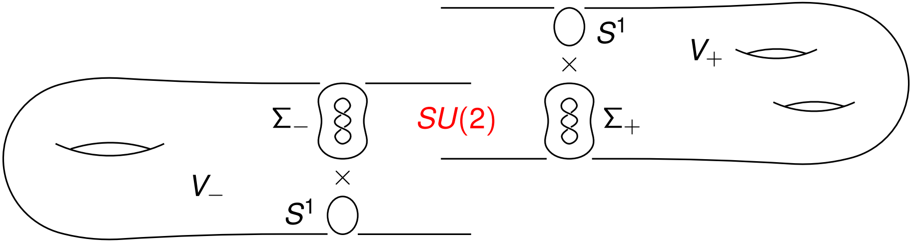
Extra twisted connected sums—Construction I

Twisted connected sums à la Kovalev and Corti-Haskins-Nordström-Pacini



Let V_+, V_- be asymptotically cylindrical Calabi-Yau threefolds

Twisted connected sums à la Kovalev and Corti-Haskins-Nordström-Pacini



Let V_+, V_- be asymptotically cylindrical Calabi-Yau threefolds with ends asymptotic to $\Sigma_\pm \times S^1 \times \mathbb{R}$, where $\Sigma_\pm$ are K3 surfaces

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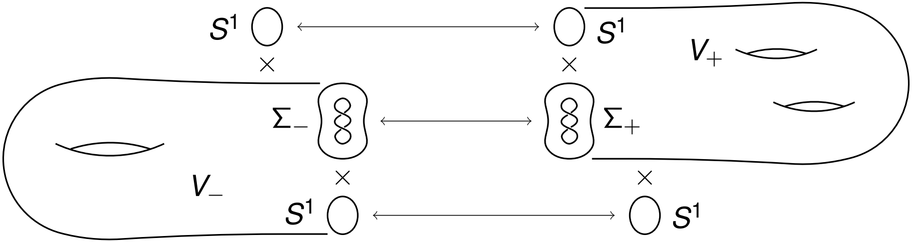

Let V_+, V_- be asymptotically cylindrical Calabi-Yau threefolds with ends asymptotic to $\Sigma_\pm \times S^1 \times \mathbb{R}$, where $\Sigma_\pm$ are K3 surfaces
Glue $V_- \times S^1$ to $V_+ \times S^1$, flipping the circles

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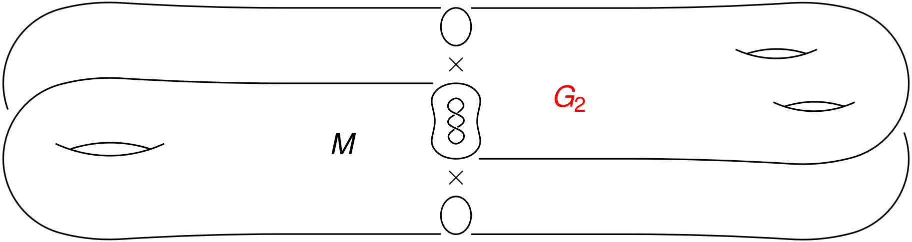
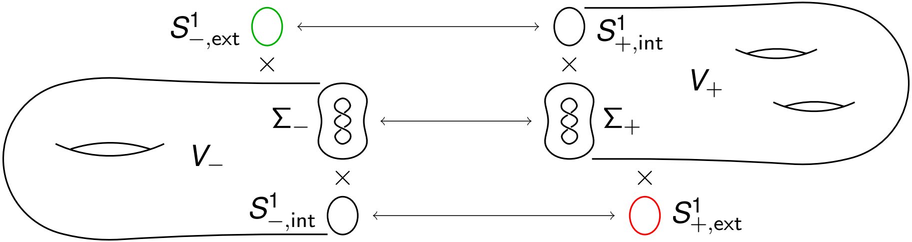
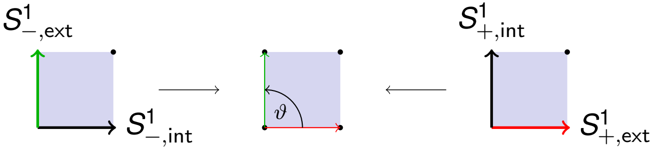


Let V_+, V_- be asymptotically cylindrical Calabi-Yau threefolds
with ends asymptotic to $\Sigma_\pm \times S^1 \times \mathbb{R}$, where $\Sigma_\pm$ are K3 surfaces
Glue $V_- \times S^1$ to $V_+ \times S^1$, flipping the circles
There is a torsion-free G_2-structure close to the one obtained by gluing

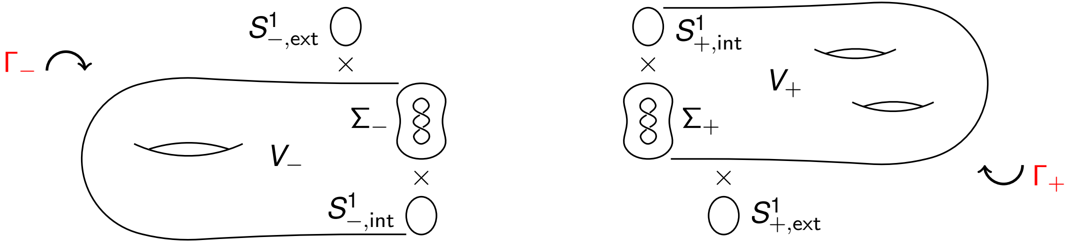
Recall twisted connected sums



Gluing of tori at angle $\vartheta = \frac{\pi}{2}$ between exterior circles



Extra twisted connected sums

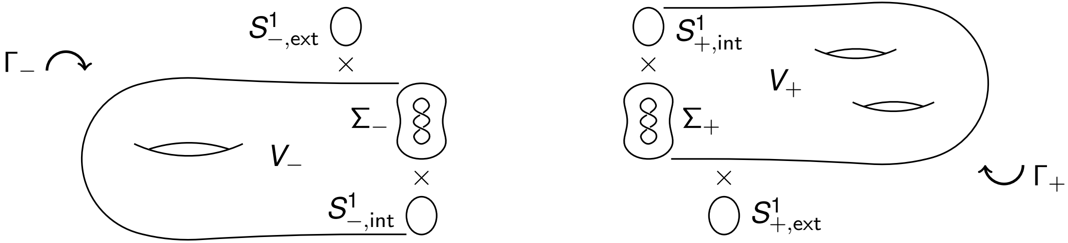


Assume that $\Gamma_\pm \cong \mathbb{Z}/k_\pm$ acts both on $V_\pm$ and on $S^1_{\pm,\mathrm{ext}}$

The induced action on $\partial V_\pm$ has to fix $\Sigma_\pm$ pointwise

The actions on $S^1_{\pm,\mathrm{int}}$ and $S^1_{\pm,\mathrm{ext}}$ have to be free

Extra twisted connected sums



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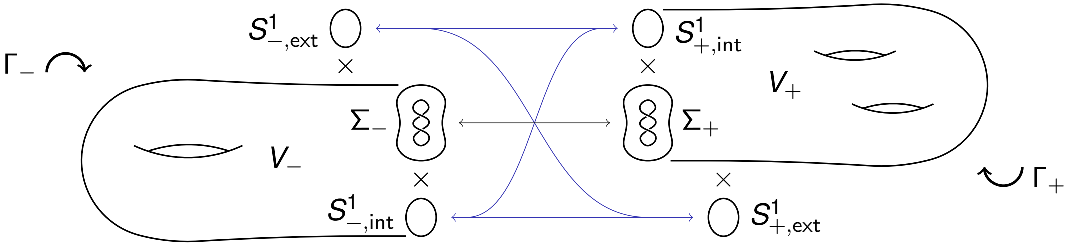
Then $T_\pm = (S^1_{\pm,\mathrm{int}} \times S^1_{\pm,\mathrm{ext}})/\Gamma_\pm$ is again a flat 2-torus

Extra twisted connected sums



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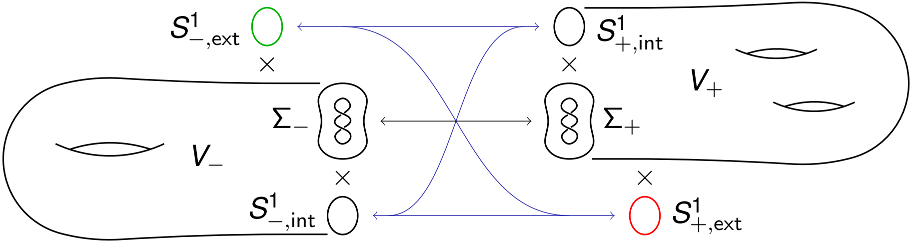
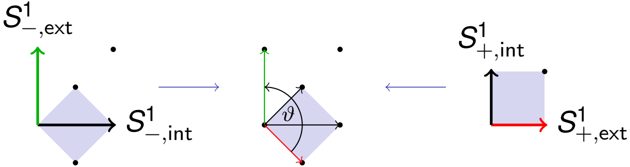
Then $T_\pm = (S^1_{\pm,\text{int}} \times S^1_{\pm,\text{ext}})/\Gamma_\pm$ is again a flat 2-torus
If both the tori and the K3 surfaces are isometric,
we can glue $M_\pm = (V_\pm \times S^1_{\pm,\text{ext}})/\Gamma_\pm$ at various angles ϑ

Extra twisted connected sums



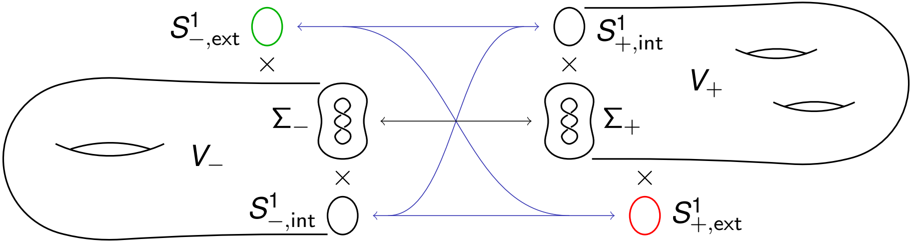
Modified gluing of tori at angle $\vartheta = \frac{3}{4}\pi$

$\Gamma_- \cong \mathbb{Z}/2$

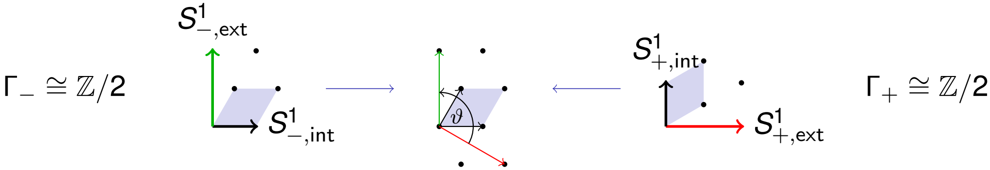


$\Gamma_+ \cong \{0\}$

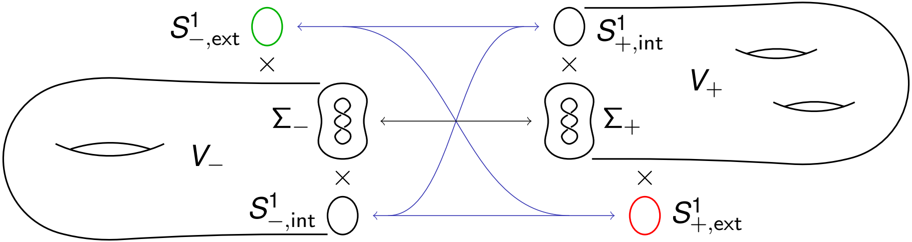
Extra twisted connected sums



Modified gluing of tori at angle $\vartheta = \frac{2}{3}\pi$

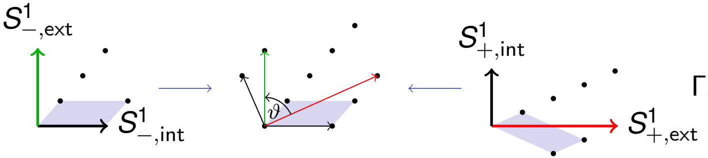


Extra twisted connected sums



Modified gluing of tori at angle $\vartheta = \arccos\left(\frac{1}{\sqrt{6}}\right) \quad \notin \mathbb{Q}\pi$

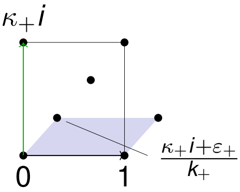
$\Gamma_- \cong \mathbb{Z}/3$



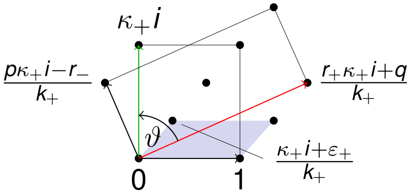
$\Gamma_+ \cong \mathbb{Z}/4$

Assume that $\Gamma_\pm \cong \mathbb{Z}/k_\pm$
acts on $\tilde{T} = S^1_{\pm,\mathrm{int}} \times \kappa_\pm S^1_{\pm,\mathrm{ext}}$
A torus matching is described by

- A number $\varepsilon_+ \in (\mathbb{Z}/k_+)^\times$ if $k_+ > 1$



Extra twisted connected sums—Matching the tori



Assume that $\Gamma_\pm \cong \mathbb{Z}/k_\pm$ acts on $\tilde{T} = S^1_{\pm,\text{int}} \times \kappa_\pm S^1_{\pm,\text{ext}}$

A torus matching is described by

- A number $\varepsilon_+ \in (\mathbb{Z}/k_+)^\times$ if $k_+ > 1$
- A gluing matrix $G = \begin{pmatrix} r_+ & p \\ q & -r_- \end{pmatrix}$, here $\begin{pmatrix} 2 & 2 \\ 5 & -1 \end{pmatrix}$ with $\det G = -k_+ k_-$ and $r_+ r_-, pq \geq 0$

satisfying some extra conditions (only finitely many choices for ε_+, G possible)

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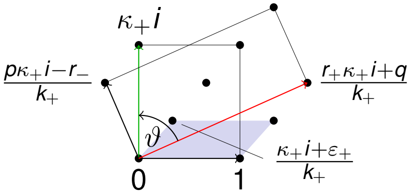

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From G, recover

- The aspect ratios $\kappa_+ = \frac{\ell(S^1_{+,\mathrm{ext}})}{\ell(S^1_{+,\mathrm{int}})} = \sqrt{\frac{qr_-}{pr_+}}$ and $\kappa_- = \frac{\ell(S^1_{-,\mathrm{ext}})}{\ell(S^1_{-,\mathrm{int}})} = \sqrt{\frac{qr_+}{pr_-}}$
- The gluing angle $\vartheta = \arg(r_+\kappa_+ + iq) \in (-\pi, \pi]$
- The fundamental group $\pi_1(M) \cong \mathbb{Z}/p$

Extra twisted connected sums—Matching the tori

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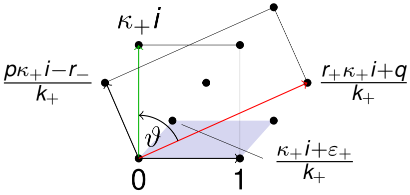

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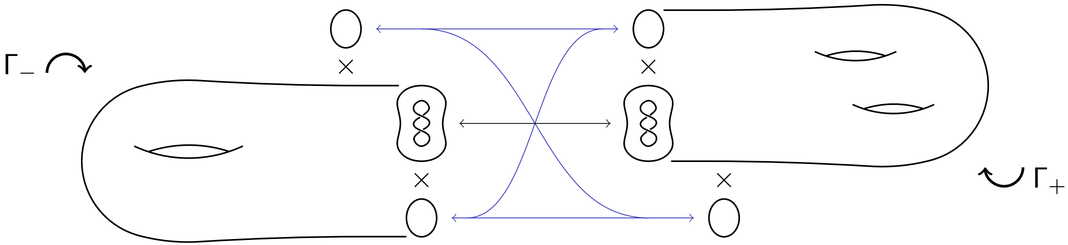
There is a similar K3 matching problem involving a hyperkähler rotation by ϑ

Computation of the ν-invariant—Strategy



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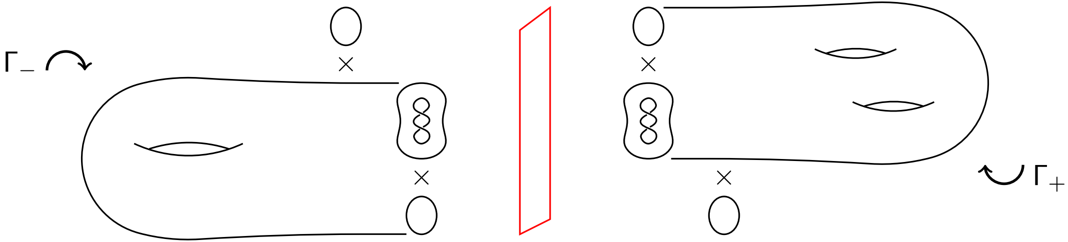

- Gluing formulas for η-Invariants [. . . , Bunke, Kirk-Lesch]

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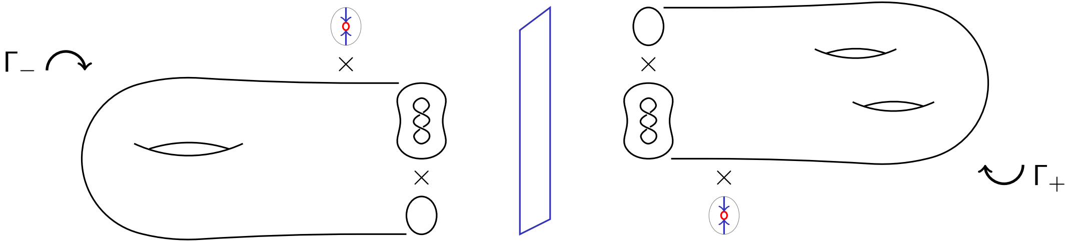

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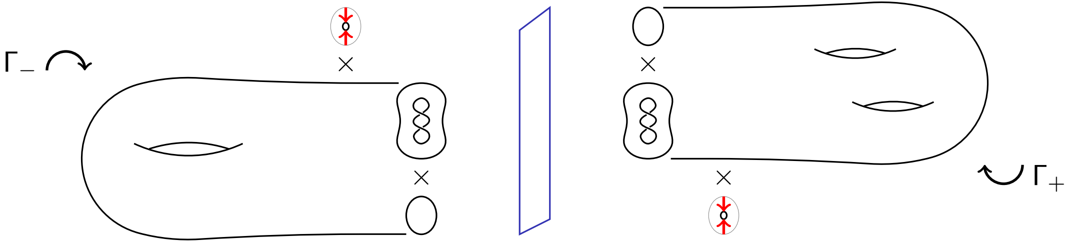

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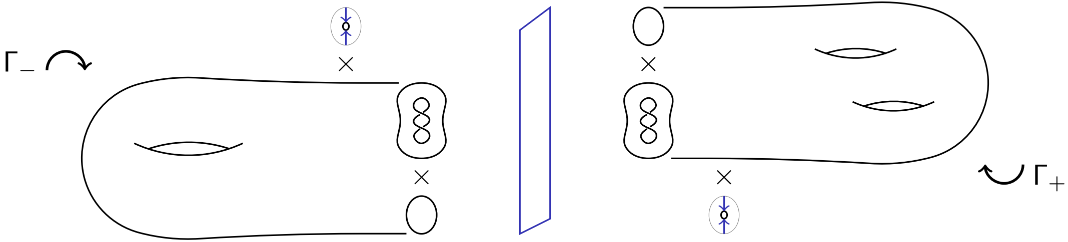

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- Variational formulas (with boundary) [Bismut-Cheeger, Dai-Freed]
- Hyperbolic geometry / analytic number theory [G-Nordström-Zagier]

Write $M = M_+ \cup_{M_0} M_-$ with $M_0 = \Sigma \times T^2$
Define Dirac operators D_{M_0} and B_{M_0} on M_0
There is a natural choice of Lagrangians $L_{B,\pm} \subset \ker(B_{M_0})$ and $L_{D,\pm} \subset \ker(D_{M_0})$

Computation of the ν-invariant—The gluing formula

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Using modified Atiyah-Patodi-Singer boundary conditions, define

$$\bar{\nu}(M_\pm, g) = 3\eta_{\mathrm{APS}}(B_{M_\pm}; L_{B,\pm}) - 24\eta_{\mathrm{APS}}(D_{M_\pm}; L_{D,\pm}) \quad \in \quad \mathbb{R}$$

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There exists $m(L_{B,+}, L_{B,-}) \in \mathbb{Z}$ depending only on the K3 matching such that

$$\bar{\nu}(M, g) = \bar{\nu}(M_+, g) + \bar{\nu}(M_-, g) + 144\frac{\vartheta}{\pi} - 72 + 3m(L_{B,+}, L_{B,-})$$

Recall that the gluing angle ϑ depends only on the gluing matrix

Compute adiabatic limit ($r \to 0$) of η-invariants on manifolds with boundary
We use the adiabatic limit theorems of [Bismut-Cheeger '92, Dai '01, G '14]

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Except over the singular set of $V_\pm/\Gamma_\pm$,
the manifold $M_\pm$ is locally isometric to a product
The adiabatic limit depends only on isolated fixpoints of elements of $\Gamma_\pm$
Write

$$D_{\gamma_\pm}(V_\pm) = \lim_{r\to 0} \bar{\nu}(M_{\pm,r})$$
$$= \frac{3}{k_\pm} \sum_{j=1}^{k_\pm - 1} \cot \frac{\pi j}{k_\pm} \sum_{p \in V_{\pm,j}} \frac{\cos\frac{\alpha_{j,1}(p)}{2} \cos\frac{\alpha_{j,2}(p)}{2} \cos\frac{\alpha_{j,3}(p)}{2} - 1}{\sin\frac{\alpha_{j,1}(p)}{2} \sin\frac{\alpha_{j,2}(p)}{2} \sin\frac{\alpha_{j,3}(p)}{2}} \quad \in \quad \mathbb{Q}$$

Shrink S^1-fibres of the flat orbibundle $M_\pm \to V_\pm/\Gamma_\pm$ to 0

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By [Cheeger '87, Bismut-Cheeger '91, Dai-Freed '94]
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- the integral of a Chern-Simons form over the interior
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Let $\tilde{\eta}(\mathbb{A})$ be the η-form of the family of tori $(S^1_{\pm,\mathrm{int}} \times rS^1_{\pm,\mathrm{ext}})/\Gamma_\pm$ for $r \in (0, \kappa_\pm)$
The boundary contribution is given by

$$\bar{\nu}(M_\pm) - \lim_{r\to 0} \bar{\nu}(M_{\pm,r}) = -\frac{144}{\pi} F_{k_\pm,\varepsilon_\pm}(\kappa_\pm) = 288 \int_0^{\kappa_\pm} \tilde{\eta}(\mathbb{A})$$

Here, $\kappa_\pm$ are the aspect ratios of the tori and depend only the gluing matrix

Computation of the ν-invariant—Hyperbolic geometry

Represent the torus $(S^1_{\pm,\mathrm{int}} \times rS^1_{\pm,\mathrm{ext}})/\Gamma_\pm$ by $\tau \in \mathcal{H} = \{\, z \in \mathbb{C} \mid \mathrm{Im}(z) > 0 \,\}$
Then $\tilde{\eta}(\mathbb{A}) \in \Omega^1(\mathcal{H})$ is $SL(2,\mathbb{Z})$-invariant and satisfies $d\tilde{\eta}(\mathbb{A}) = -\frac{1}{4\pi}\, dA_{\mathrm{hyp}}$
by the family index theorem [Bismut '86, Bismut-Cheeger '92]

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Idea. Use hyperbolic geometry
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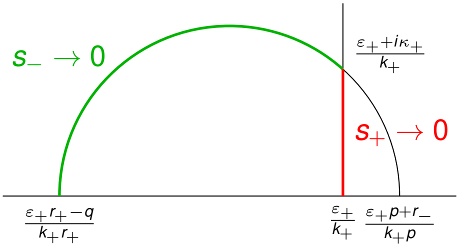
Adiabatic limits—geodesic rays



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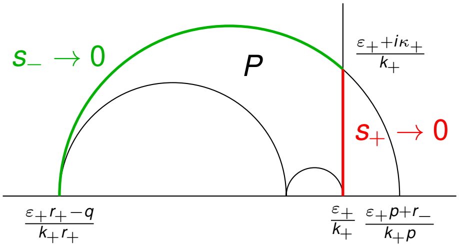
Adiabatic limits—geodesic rays

$\tilde\eta(\mathbb{A}) = 0$ along rectangular families
Determine P using continued fractions



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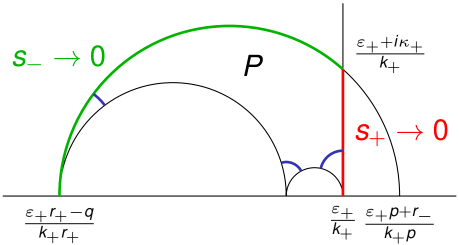
Cusps—families of adiabatic limits
[Bunke-Ma '04, G-Nordström]



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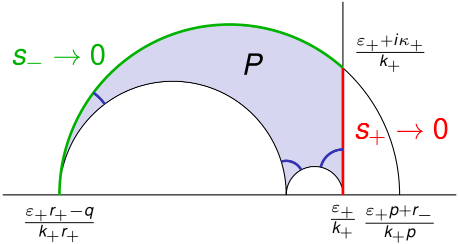
Use the hyperbolic area formula for P



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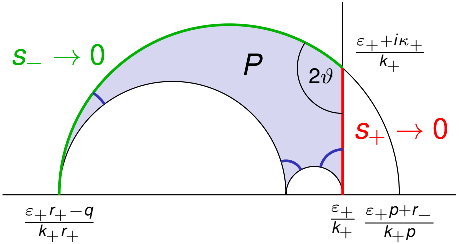
Cusps—families of adiabatic limits
[Bunke-Ma '04, G-Nordström]



Use the hyperbolic area formula for P

The cusp contributions add up to a classical Dedekind sum [Zagier '75]
The angle 2ϑ at the finite corner cancels $144\frac{\vartheta}{\pi}$ in the gluing formula

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Compute the sum of the variational terms using the functional equations

$$\mathrm{H}(\tau+1) = \frac{\pi i}{12} + \mathrm{H}(\tau) \qquad \text{and} \qquad \mathrm{H}\left(-\frac{1}{\tau}\right) = \frac{1}{2} \log \frac{\tau}{i} + \mathrm{H}(\tau)$$

The correction terms add up to give a classical Dedekind sum

Let M be an extra twisted connected sum
with gluing matrix $G = \begin{pmatrix} r_+ & p \\ q & -r_- \end{pmatrix}$ and gluing angle ϑ
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For integers $n > 0$ and a define the classical Dedekind sum

$$S(a,n) = \sum_{j=1}^{n-1} \left(\left(\frac{j}{n}\right)\right)\left(\left(\frac{aj}{n}\right)\right) \in \frac{1}{6n}\mathbb{Z} \quad \text{with} \quad ((x)) = \begin{cases} 0 & x \in \mathbb{Z} \\ x - \lfloor x \rfloor - \frac{1}{2} & x \notin \mathbb{Z} \end{cases}$$

Computation of the ν-invariant—Final formula

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Theorem (G-Nordström)

Assume $q > 0$. Then $a = \frac{r_+ - \varepsilon_+^* q}{k_+} \in \mathbb{Z}$ and

$$\bar{\nu}(M,g) = D_{\gamma_+}(V_+) + D_{\gamma_-}(V_-) + 3m(L_{B_+}, L_{B_-}) + 24\left(12\, S(a,q) - \frac{r_-}{k_- q} - \frac{r_+}{k_+ q}\right)$$

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- Are there analogous invariants for singular G_2-spaces?

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Bad candidates: $\bar{\nu}(M_\pm, g_\pm) \in \mathbb{R}$

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Bad candidates: $\bar{\nu}(M_\pm, g_\pm) \in \mathbb{R}$
Better maybe: $D_{\gamma_+}(V_+) - 24\,\frac{\varepsilon_+^*}{k_+}$, $D_{\gamma_-}(V_-) - 24\,\frac{\varepsilon_-^*}{k_-} \in \mathbb{Z}$
But $\varepsilon_\pm^*$ is not well defined—and what does this mean?

Thanks for your attention!