Determinant of the weighted Laplacian for the sequence of asymptotically large compact Riemannian surfaces

Jay Jorgenson, Lejla Smajlović, Polyxeni Spilioti

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Can we give a meaning to

$$\det^*(\mathcal{A}) = \prod_{\lambda_j \neq 0} \lambda_j?$$

Regularized determinant - discrete spectrum

If the sequence $\lambda_0 \leq \lambda_1 \leq \ldots \to \infty$ grows sufficiently fast, for $\mathrm{Re}(s) \gg 0$ define (formally) the spectral zeta function

$$\zeta_{\mathcal{A}}(s) = \mathrm{tr}(\mathcal{A}^{-s}) = \sum_j{}^* \frac{1}{\lambda_j^s}.$$

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Therefore (again, formally)

$$\det{}^*(\mathcal{A}) = \exp(-\zeta'_{\mathcal{A}}(0)) = \prod_j{}^* \lambda_j.$$

Amusing example: $\infty!$

Assume $\lambda_j = j$, for $j \geq 0$. Then, the spectral zeta function is

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The product of eigenvalues is

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The well-known formula stemming from the meromorphic continuation of the Riemann zeta function and its functional equation:

$$\zeta'(0) = -\log\sqrt{2\pi},$$

hence

$$"\infty! = e^{\log\sqrt{2\pi}} = \sqrt{2\pi}."$$

Determinant of Laplacian on a flat torus

Let $T = \mathbb{R}^2 \backslash (\mathbb{Z} \oplus \tau\mathbb{Z})$, $\tau_2 =$Im$(\tau) > 0$ be a flat two-dimensional torus

The Laplacian is

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Eigenfunctions are exponentials $exp(2\pi i(nx + my/\tau_2))$, $m, n \in \mathbb{Z}$ and non-zero eigenvalues are

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Spectral zeta function:

$$\zeta_{\Delta_T} = (4\pi^2)^{-s} \sum_{(m,n)\neq(0,0)} \frac{\tau_2^s}{|m + n\tau|^{2s}} = (4\pi^2)^{-s} E_\infty(\tau, s)$$

Determinant of Laplacian on a flat torus, cont'

$$E_\infty(\tau, s) = \sum_{\gamma \in \Gamma_\infty \backslash \mathrm{PSL}(2,\mathbb{Z})} (\mathrm{Im}\gamma\tau)^s$$

the Eisenstein series associated to the modular group $\mathrm{PSL}(2, \mathbb{Z})$.

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Using the functional equation

$$E_\infty(\tau, 1-s) = \frac{\xi(2s-1)}{\xi(2s)} E_\infty(\tau, s)$$

combined with the Laurent series expansion of $E_\infty(\tau, s)$ at $s = 1$ we get

$$\det \Delta_T(\tau) = 2\pi\tau_2^{1/2}|\eta(\tau)|^2,$$

where

$$\eta(\tau) = e^{i\pi\tau/12} \prod_{n=1}^{\infty} (1 - e^{2\pi i n\tau})$$

is the Dedekind eta function.

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Promising means by which one could study modular forms using spectral data.

$\det \Delta$ is related to other zeta functions connecting spectral data with geometry and arithmetics.

Fuchsian groups

$\Gamma < \mathrm{SL}_2(\mathbb{R})$ a discrete subgroup acting totally discontinuously on the upper half plane, such that $-I_2 \in \Gamma$

$\overline{\Gamma}$ projection of Γ into $\mathrm{PSL}(2,\mathbb{R})$

$X = \overline{\Gamma}\backslash\mathbb{H}$, compact of genus $g > 1$

The volume of X is

$$\mathrm{vol}(X) = 2\pi(2g-2).$$

Unitary multiplier systems

Unitary multiplier system χ of dimension m and admissible weight $2k$ is a map from Γ to the ring $\mathcal{U}(V)$ of unitary endomorphisms of an m-dimensional Hermitian vector space V, which satisfies the following properties:

$$\chi(-I_2) = e^{-2\pi i k} I_V,$$

For γ, η in Γ, we have that

$$\chi(\gamma\eta) = \sigma_{2k}(\gamma,\eta)\chi(\gamma)\chi(\eta),$$

where σ_{2k} is a weight $2k$ factor system, meaning that for

$$\eta = \begin{pmatrix} \eta_1 & \eta_2 \\ \eta_3 & \eta_4 \end{pmatrix}, \quad \gamma = \begin{pmatrix} \gamma_1 & \gamma_2 \\ \gamma_3 & \gamma_4 \end{pmatrix}, \quad \gamma\eta = \begin{pmatrix} \delta_1 & \delta_2 \\ \delta_3 & \delta_4 \end{pmatrix}.$$

we have that

$$\sigma_{2k}(\gamma,\eta) = \exp(2\pi i k w(\gamma,\eta))$$

where

$$w(\gamma,\eta) = \arg(\gamma_3\eta z + \gamma_4) + \arg(\eta_3 z + \eta_4) - \arg(\delta_3 z + \delta_4) \in \{-1,0,1\}$$

is independent of z.

Unitary multiplier systems cont'

The weight $2k$ is *admissible* if there exists a multiplier system with weight $2k$.

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The admissible values of k are the real numbers which lie in the set

$$A_m(\Gamma) := \frac{1}{m} \frac{2\pi}{\mathrm{vol}(X)} \mathbb{Z} = \frac{\mathbb{Z}}{2m(g-1)}.$$

Trivially, for any $a \in \mathbb{Z}$, $\frac{a}{m(g-1)}$ is an admissible weight for a unitary multiplier system of dimension m.

Weighted Laplacian

The weight $2k$ Maass-Laplacian is defined as

$$\Delta_{2k} = -y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) + 2kiy \frac{\partial}{\partial x}.$$

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Δ_{2k} acts on the space of twice continuously differentiable functions $f : \mathbb{H} \to V$ that satisfy the transformation property

$$f(\gamma z) = \exp(2ik \arg(cz + d))\chi(\gamma)f(z) \quad \text{for all } \gamma = \begin{pmatrix} * & * \\ c & d \end{pmatrix} \in \Gamma.$$

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Spectrum of Δ_{2k}: discrete with eigenvalues

$$\lambda_0 = |k|(1 - |k|) \leq \lambda_1 \leq \cdots \leq \lambda_j = s_j(1 - s_j) \leq \cdots \to +\infty$$

Weighted Laplacian - small detour

$\lambda_0 = 0$ iff $k \in \{-1, 0, 1\}$.

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Case $k = \pm 1$: the subspace of L^2 with the eigenvalue 0 is isomorphic to the Hilbert space of classical cusp forms of weight 2

One can move between even weight subspaces using rasing and lowering operators (Fay); very useful for the study of automorphic forms.

The Selberg zeta function

The *twisted* Selberg zeta function is defined for $\mathrm{Re}(s) > 1$ by

$$Z(s;\chi) = \prod_{\substack{P_0,\\ \mathrm{tr}(P_0)>2}} \prod_{\ell=0}^{\infty} \det\left(I_m - \chi(P_0)N(P_0)^{-s-\ell}\right); \tag{1}$$

the product runs through all primitive hyperbolic elements P_0 of the group Γ, and $N(P_0)$ is the norm of the element P_0.

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Zeros at $\frac{1}{2} \pm ir_j$, where $s_j = \frac{1}{2} + ir_j$.

Hence, $Z(1;\chi) = 0$ iff $\lambda_0 = 0$ iff $k \in \{-1, 0, 1\}$.

Zeta regularized determinant

For $\mathrm{Re}(s) > 1$ and $\mathrm{Re}(w)$ sufficiently large define the spectral zeta function by

$$\zeta(w, s; \chi) = \sum_{j \geq 0} (s_j(1 - s_j) - s(1 - s))^{-w}.$$

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For $\mathrm{Re}(s) > 1$ and $|k| - s \notin \mathbb{Z}_{\geq 0}$ regularized det is defined as

$$\det(\Delta_{\chi,2k} - s(1 - s)) = e^{-\frac{\partial}{\partial w} \zeta(w,s;\chi)|_{w=0}}.$$

(Gong 1995) For $\mathrm{Re}(s) > 1$ and $|k| - s \notin \mathbb{Z}_{\geq 0}$

$$\det(\Delta_{\chi,2k} - s(1 - s)) = Z(s; \chi) Z_I(s; \chi) e^{\tilde{c}},$$

$$Z_I(s;\chi) = \exp\left\{ \frac{m\mathrm{vol}(X)}{2\pi} \left(s\log(2\pi) + s(1-s) + \left(\frac{1}{2}+k\right)\log\Gamma\left(s+k\right) + \left(\frac{1}{2}-k\right)\log\Gamma\left(s-k\right) - \log G\left(s+k+1\right) - \log G\left(s-k+1\right) \right) \right.$$

and

$$\widetilde{c} = -m\frac{\mathrm{vol}(X)}{2\pi}\left(\log\sqrt{2\pi} + \frac{1}{4} - 2\zeta'(-1)\right)$$

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When $k \in (0,1)$ zero is not an eigenvalue of $\Delta_{\chi,2k}$ hence

$$\det\left(\Delta_{\chi,2k}\right) = Z(1;\chi)Z_I(1;\chi)e^{\tilde{c}}.$$

Natural question

What can we say about asymptotics of $\log \det \Delta_{\chi,2k}$ associated to a sequence $\{X_n\}$ of random compact hyperbolic surfaces such that $\mathrm{vol}(X_n) \to \infty$?

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When $\{X_n\}$ runs through "typical" random compact surfaces, Frédéric Naud (2023) proved that

$$\frac{\log \det \Delta_0}{\mathrm{vol}(X_n)} \to \frac{1}{2\pi}\left(2\zeta'(-1) - \frac{1}{4} + \log \sqrt{2\pi}\right), \text{ as } n \to \infty.$$

Similar "folklore" results are known for asymptotics through congruence covers.

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- surfaces in the sequence belong to one of "standard" models of compact hyperbolic surfaces with volume tending to infinity (e.g. Brooks-Makover model, Weil-Petersson or random cover model)
- There is a uniform spectral gap
- There is a bound on the number of closed geodesics up to certain length (satisfied with probability 1 in all three models)

Setup

We consider a sequence $\{(X_n, \chi_n)\}$ of:

- compact hyperbolic surfaces such that $\mathrm{vol}(X_n) = 4\pi(g_n - 1) \to \infty$, $n \to \infty$

- $m \times m$ multiplier systems χ_n of admissible weights $k_n \in (0,1)$ such that

$$\lim_{n\to\infty} k_n = \alpha \in [0,1).$$

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Additionally - $\mathcal{N}_{k,n}$ denotes the number of small eigenvalues of Δ_{2k_n}

Assumptions: weak spectral gap

(i) (Weak spectral gap) Assume that

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If, for example $\mathcal{N}_{k,n} \leq \mathcal{M}$ uniformly in n, then from "admissibility" of k_n, weak spectral gap holds true.

Assumptions: weak spectral gap

(i) (Weak spectral gap) Assume that

$$\limsup_{n\to\infty} \frac{\mathcal{N}_{k,n}}{\lambda_{1,n}\mathrm{vol}(X_n)} = \beta < \infty \tag{2}$$

Why weak spectral gap?

By a crude bound:

$$\mathcal{N}_{k,n} \leq Cg_n$$

the spectral gap yields weak spectral gap.

If, for example $\mathcal{N}_{k,n} \leq \mathcal{M}$ uniformly in n, then from "admissibility" of k_n, weak spectral gap holds true.

Assumptions: weak spectral gap cont'

Spectral gap holds true for "typical" compact surface in three models of random surfaces: Brooks–Makover, random covers and Weil–Petersson.

Even more is true in Weil–Petersson model - an optimal $1/4 - \epsilon$ *expected* gap is deduced - preprint by Anantharaman and Monk (2025)

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Our aim: deterministic result.

Example by Sarnak and Buser: Benjamini–Schramm convergence of the sequence of compact surfaces $\{X_n\}$ of genus g_n does not necessarily imply spectral gap.

There exists a Benjamini-Schramm convergent sequence (injectivity radius $\to \infty$) with the first eigenvalues $\lambda_{1,n}$ bounded by a constant times $\log g_n / g_n$.

Assumptions: uniform discreteness

(ii)(Uniform discreteness) Let $1 < \delta_n = \min_{P \in \Gamma_n,\, P \neq \mathrm{Id}} \{N(P)\}$. Then, there exists $\eta > 1$ such that $\delta_n \geq \eta$ for all $n \geq 1$.

Equivalent to assuming that $\mathrm{sys}(X_n) \geq \log \eta > 0$ for all n

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Necessary to deduce that if a sequence $\{X_n\}$ of compact surfaces converges in Benjamini-Schramm sense to $\mathbb{H}$, then the spectral density (normalized by volume) of the Laplacian on X_n converges to the spectral measure of the Laplacian on $\mathbb{H}$ (Le Masson and Sahlsten; Monk)

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Geometrically - we stay away from the boundary in the moduli space of compact surfaces.

Assumptions - geodesic count

We introduce the following definition: The sequence $\{X_n\}_{n\geq 1}$ is said to satisfy the hypothesis $\mathcal{H}(C, L, \kappa)$ for some $C, L > 0$ and $0 < \kappa < 1$ if

$$N_{X_n}(L) := |\{P \in \Gamma_n : N(P) \leq L\}| \leq C\mathrm{vol}(X_n)^{\kappa}.$$

Naud proved that hypothesis $\mathcal{H}(C, L, \kappa)$ holds with probability one in all three models of random surfaces discussed here.

Main result: asymptotics of $\log \det(\Delta_{2k_n})$

$\{X_n\}$ - the sequence of Riemann surfaces, $\{\chi_n\}$ the corresponding sequence of $m \times m$ unitary multiplier systems of weights $2k_n \in (0,2)$ such that $\lim_{n\to\infty} k_n = \alpha \in [0,1)$. Assume weak spectral gap and uniform discreteness.

If $\alpha = 0$, assume that the multiplicities $m(\lambda_{0,n})$ of the first eigenvalue of the Laplacian Δ_{2k_n} are bounded uniformly in n.

Then, for all $\varepsilon > 0$ there exists $L_\varepsilon > 0$ such that, assuming $\mathcal{H}(C, L_\varepsilon, \kappa)$, for some constant C, uniformly for $\mathrm{vol}(X_n)$ large we have

$$\left| \frac{2\pi \log \det(\Delta_{2k_n})}{m\mathrm{Vol}(X_n)} - \left(2\zeta'(-1) + \log\sqrt{2\pi} - \frac{1}{4} + \left(\frac{1}{2} + \alpha\right) \log \Gamma\left(1+\alpha\right) + \right.\right.$$
$$\left.\left. + \left(\frac{1}{2} - \alpha\right) \log \Gamma\left(1-\alpha\right) - \log G\left(2+\alpha\right) - \log G\left(2-\alpha\right) \right) \right| < \varepsilon.$$

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Note: this agrees with deterministic result of Naud for $\alpha = 0$; our assumptions and proof are different.

Main result: asymptotics of $\log \det(\Delta_{2k_n})$, *example*

Fix a rational number $k = \frac{p}{q} \in (0,1)$, p, q coprime.

$\{X_n\}$ - sequence of Riemann surfaces of genus $g_n = \ell_n q + 1$, where $\ell_n \uparrow \infty$.

$k = \frac{2mp\ell_n}{2m(g_n-1)} \in \frac{\mathbb{Z}}{2m(g_n-1)}$ is admissible, hence there is $m \times m$ unitary multiplier system $\{\chi_n\}$ of weight k. Spectral gap immediately fulfilled.

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Assuming uniform discreteness for $\{X_n\}$, we have

$$\frac{2\pi \log \det(\Delta_{2p/q})}{m\mathrm{Vol}(X_n)} \sim 2\zeta'(-1) + \log\sqrt{2\pi} - \frac{1}{4} + \left(\frac{1}{2} + \frac{p}{q}\right)\log\Gamma\left(1+\frac{p}{q}\right) + \\ + \left(\frac{1}{2} - \frac{p}{q}\right)\log\Gamma\left(1-\frac{p}{q}\right) - \log G\left(2+\frac{p}{q}\right) - \log G\left(2-\frac{p}{q}\right).$$

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When $k_n = 1/2$ for all n:

$$\frac{2\pi \log \det(\Delta_1)}{m\mathrm{Vol}(X_n)} \sim \frac{5}{12}\log 2 - \zeta'(-1) - \frac{1}{4}.$$

Main result - counting pseudoprimes

Want to use $\det(\Delta_{\chi,2k}) = Z(1;\chi)Z_I(1;\chi)e^{\tilde{c}}$.

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Hejhal proved that for $\lambda_{j,n} = s_{j,n}(1-s_{j,n})$ and $x \geq 2$

$$\Psi(x,\chi_n) = \sum_{\lambda_{j,n}\leq 1/4} m(\lambda_{j,n})\frac{x^{s_{j,n}}}{s_{j,n}} + O_{X_n,\chi_n}(x^{3/4})$$

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Problem: the implied constant depends upon X_n. How large can it be when g_n is large??

Main result - uniform, generalized PGTs

When $\chi_n = \mathrm{Id}_1$, an explicit upper bound for the constant in PGT (called Huber's constant) is known when X is not necessarily compact (Friedman, Jorgenson, Kramer; improved by Avdispahić).

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Yunhui Wu and Yuhao Xue - proved the PGT with 5/6 instead of 3/4 in the exponent, but with an explicit bound for the constant in case $\chi_n = \mathrm{Id}_1$:

$$\sum_{P \text{ primitive: } N(P) \le x} 1 = \sum_{\lambda_j \le 1/4} m(\lambda_j)\mathrm{Li}(x^{s_j}) + \\ + O\left(\frac{g}{\log x} x^{2/3} \left(x^{1/6} + \max\left\{ 0, -\log(\mathrm{sys}(X)) - \frac{1}{6}\log x \right\} \right) \right)$$

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Note: this is count of prime geodesic. We need *weighted* count of *all* geodesics.

Main result - uniform, generalized PGT1

For $x \geq 2$, fixed Riemann surface X of genus g and $m \times m$ unitary multiplier system χ of weight k we have that

$$\Psi(x,\chi) = \sum_{\lambda_j \leq 1/4} m(\lambda_j) \frac{x^{s_j}}{s_j} + g(X,\chi) x^{3/4}$$

where $m(\lambda_j)$ denotes the multiplicity of the small eigenvalue λ_j and

$$|g(X,\chi)| \leq C_1 \frac{m(g-1)}{\mathrm{sys}(X)} + (\mathcal{N}_k - 1)\left(\frac{C_2}{1-s_1} + C_3\right) + (\mathcal{N}_0 - 1)\left(\frac{C_4}{1-\widetilde{s}_1} + C_5\right),$$

for some absolute constants C_1, C_2, C_3, C_4, C_5 independent of X and χ.

Here $\mathcal{N}_k$ is the number of small eigenvalues of Δ_{2k} and $\mathcal{N}_0$ is the number of small eigenvalues of Δ_0.

$\lambda_1 = s_1(1-s_1)$ and $\widetilde{\lambda}_1 = \widetilde{s}_1(1-\widetilde{s}_1)$

Main result - uniform, generalized PGT2

For $x \geq x(g)$, where $x(g)$ is a constant depending upon the genus we have that

$$\Psi(x,\chi) = \sum_{\lambda_j \leq 1/4} m(\lambda_j) \frac{x^{s_j}}{s_j} + \mathcal{G}(x; X, \chi)$$

where

$$|\mathcal{G}(x; X, \chi)| \leq \mathcal{C}_1 mgx^{5/6} + \mathcal{C}_2 mgx^{2/3} \max\left\{0, \log\left(\frac{1}{\mathrm{sys}(X) x^{1/6}}\right)\right\},$$

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Natural question: comparison of generalized PGT1 and PGT2

$\mathcal{N}_k$, $\mathcal{N}_0$ bounded by Cg; however $\widetilde{\lambda}_1$ can be close to 0, so PGT1 error not optimal in the eigenvalue aspect, though the power of x is better.

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Step 2: Key lemma (Hejhal)

$$\Psi(x, \chi) = \sum_{P \in \Gamma,\, \mathrm{Tr}(P) > 2 \colon N(P) \le x} \Lambda(P) \mathrm{Re}\left(\mathrm{Tr}(\chi(P))\right).$$

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Therefore, for $c \in (0, 1/m)$

$$\Psi(x,\chi) = \frac{1}{c}\left[(\Psi(x,\mathrm{Id}_1) + c\Psi(x,\chi)) - \Psi(x,\mathrm{Id}_1)\right]$$

and $\Psi(x,\mathrm{Id}_1) + c\Psi(x,\chi)$ is non-decreasing so methods of Step 1 apply.

Proof: asymptotics of $\det \Delta_{2k_n}$

Start with $\log \det(\Delta_{2k_n} - s(1-s)) = \log Z(s;\chi_n) + \log Z_I(s;\chi_n) + \tilde{c}$.

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Left to prove: for fixed $\epsilon > 0$ there exists L_ϵ such that assuming $\mathcal{H}(C, L_\epsilon, \kappa)$ for come $C > 0$, $\kappa \in (0,1)$

$$\left| \lim_{s\to 1^+} \frac{2\pi \log Z(s;\chi_n)}{m\mathrm{Vol}(X_n)} \right| < \frac{\varepsilon}{2} \qquad (3)$$

holds true for large enough n

$$\lim_{s\to 1^+} \log Z(s;\chi_n) = -\int\limits_{\delta_n-\delta}^{\infty} \frac{\Psi(t,\chi_n)(\log t+1)}{t^2\log^2 t}dt = -\left(\int\limits_{L}^{\infty}+\int\limits_{\delta_n-\delta}^{L}\right).$$

Look at sum of two integrals. By PGT2:

$$\frac{1}{\mathrm{vol}(X_n)}\left|\int\limits_{L}^{\infty}\frac{\Psi(t,\chi_n)(\log t+1)}{t^2\log^2 t}dt\right| \leq \frac{2L^{-\lambda_{0,n}}}{\log L}\left[\frac{m(\lambda_{0,n})}{\lambda_{0,n}\mathrm{vol}(X_n)}+\frac{\mathcal{N}_{k,n}}{\lambda_{1,n}\mathrm{vol}(X_n)}\right] + \frac{1}{L^{\frac{1}{6}}\log L}\frac{C(m,\eta)g_n}{\mathrm{vol}(X_n)},$$

Use

$$\lambda_{0,n}\mathrm{vol}(X_n) = \frac{2\pi a_n}{m}\left(1-\frac{a_n}{2m(g_n-1)}\right) \geq \frac{2\pi}{m}\left(1-\frac{a_n}{2m(g_n-1)}\right)$$

and deduce that the first integral is bounded by $\frac{\varepsilon}{16\pi}$ for $L > L_\epsilon$

Second integral: apply $\mathcal{H}(C, L_\varepsilon, \kappa)$ for some $C > 0$:

$$\left| \int\limits_{\delta_n-\delta}^{L_\varepsilon} \frac{\Psi(t,\chi_n)(\log t + 1)}{t^2 \log^2 t} dt \right| \leq m \int\limits_{\delta_n-\delta}^{L_\varepsilon} \frac{(\log t + 1) \sum\limits_{P\in\Gamma_n;\, N(P)\leq t} \Lambda(P)}{t^2 \log^2 t} dt$$

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under hypothesis $\mathcal{H}(C, L_\varepsilon, \kappa)$ we have

$$\sum_{N(P)\leq L_\varepsilon} \Lambda(P) \leq \sum_{P\in\Gamma_n;\, N(P)\leq L_\varepsilon} \log N(P_0) \leq \log L_\varepsilon N_{X_n}(L_\varepsilon) \leq C \log L_\varepsilon (\mathrm{vol}(X_n)^\kappa),$$

hence dividing by the volume

$$\frac{1}{m(2g_n-2)} \left| \int_{\delta_n-\delta}^{L_\varepsilon} \frac{\Psi(t,\chi_n)(\log t - 1)}{t^2 \log^2 t} dt \right| \leq \frac{2\pi CB \log L_\varepsilon}{\mathrm{vol}(X_n)^{1-\kappa}}.$$

The end

Thank you for your attention!