The q-Onsager algebra and its finite-dimensional irreducible modules

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Overview

This talk is about the q-**Onsager algebra** O_q.

We will describe the finite-dimensional irreducible O_q-modules that satisfy a mild assumption.

Our description involves the concept of a **tridiagonal pair**.

As a warmup, let us recall this concept.

Preliminaries

Let $\mathbb{F}$ denote a field.

Let V denote a nonzero, finite-dimensional vector space over $\mathbb{F}$.

Let the algebra $\mathrm{End}(V)$ consist of the $\mathbb{F}$-linear maps from V to V.

Pick any $A, A^* \in \mathrm{End}(V)$.

The definition of a tridiagonal pair

Definition (Ito+Tanabe+Ter, 2001)

We call A, A^* a **tridiagonal pair on** V whenever:

(i) each of A, A^* is diagonalizable;

(ii) there exists an ordering $\{V_i\}_{i=0}^{d}$ of the eigenspaces of A such that

$$A^* V_i \subseteq V_{i-1} + V_i + V_{i+1} \qquad (0 \leq i \leq d),$$

where $V_{-1} = 0$ and $V_{d+1} = 0$;

(iii) there exists an ordering $\{V_i^*\}_{i=0}^{\delta}$ of the eigenspaces of A^* such that

$$A V_i^* \subseteq V_{i-1}^* + V_i^* + V_{i+1}^* \qquad (0 \leq i \leq \delta),$$

where $V_{-1}^* = 0$ and $V_{\delta+1}^* = 0$;

(iv) $\nexists$ a subspace $W \subseteq V$ such that $AW \subseteq W$, $A^* W \subseteq W$, $W \neq 0$, $W \neq V$.

The q-**Onsager algebra** O_q is defined as follows.

Fix $0 \neq q \in \mathbb{F}$ such that $q^4 \neq 1$.

Recall the commutator

$$[R, S] = RS - SR$$

and q-commutator

$$[R, S]_q = qRS - q^{-1}SR.$$

The definition of O_q

Definition (Ter 2003, Baseilhac 2005)

The q-**Onsager algebra** O_q is defined by generators A, A^* and relations

$$[A,[A,[A,A^*]_q]_{q^{-1}}] = (q^2-q^{-2})^2[A^*,A],$$
$$[A^*,[A^*,[A^*,A]_q]_{q^{-1}}] = (q^2-q^{-2})^2[A,A^*].$$

The above relations are called the q-**Dolan/Grady relations**.

Comments on the normalization

We comment on the normalization.

On the RHS we see the coefficients $(q^2 - q^{-2})^2$.

These coefficients are chosen for convenience, and are not important.

We can adjust the coefficients by multiplying each of A, A^* by an arbitrary nonzero scalar.

More generally, consider the affine change of variables

$$A \mapsto rA + sI, \qquad\qquad A^* \mapsto r^*A^* + s^*I$$

with nonzero $r, r^* \in \mathbb{F}$ and $s, s^* \in \mathbb{F}$.

Comments on the normalization, cont.

After this change of variables, the q-Dolan/Grady relations become

$$0 = [A, A^2A^* - \beta AA^*A + A^*A^2 - \gamma(AA^* + A^*A) - \varrho A^*],$$
$$0 = [A^*, A^{*2}A - \beta A^*AA^* + AA^{*2} - \gamma^*(A^*A + AA^*) - \varrho^* A]$$

where

$$\beta = q^2 + q^{-2},$$
$$\gamma = -s(q - q^{-1})^2, \qquad \gamma^* = -s^*(q - q^{-1})^2,$$
$$\varrho = s^2(q - q^{-1})^2 - r^2(q^2 - q^{-2})^2,$$
$$\varrho^* = s^{*2}(q - q^{-1})^2 - r^{*2}(q^2 - q^{-2})^2.$$

The tridiagonal relations

The relations

$$0 = [A, A^2A^* - \beta AA^*A + A^*A^2 - \gamma(AA^* + A^*A) - \varrho A^*],$$
$$0 = [A^*, A^{*2}A - \beta A^*AA^* + AA^{*2} - \gamma^*(A^*A + AA^*) - \varrho^* A]$$

are called the **tridiagonal relations**, and they are satisfied by any tridiagonal pair.

The tridiagonal relations first appeared in **Algebraic Graph Theory**; see the paper

P. Terwilliger. The subconstituent algebra of an association scheme, III. J. Alg. Combin. 2 (1993) 177–210.

Our goal for this talk:

Describe the **finite-dimensional, irreducible O_q-modules V** under the following assumptions:

- $\mathbb{F}$ is algebraically closed;
- q is not a root of unity;
- the O_q-generators A, A^* are diagonalizable on V;
- $\dim V \geq 2$.

These assumptions are in force for the rest of the talk.

STEP I: Show that the O_q-generators A, A^* act on the O_q-module V as a tridiagonal pair.

The set $\mathcal{D}$

Let $\mathcal{D}$ denote the set of eigenvalues for A on V.

By construction, $\mathcal{D}$ is nonempty.

For $\lambda \in \mathcal{D}$ let $E_\lambda \in \mathrm{End}(V)$ denote the projection onto the λ-eigenspace for A on V.

E_λ is often called the **primitive idempotent** for A and the eigenvalue λ.

The primitive idempotents for A on V

By linear algebra, the following hold on V:

(i) $E_\lambda E_\mu = \delta_{\lambda,\mu} E_\lambda \ (\lambda, \mu \in \mathcal{D})$;

(ii) $I = \sum_{\lambda \in \mathcal{D}} E_\lambda$;

(iii) $AE_\lambda = \lambda E_\lambda = E_\lambda A \ (\lambda \in \mathcal{D})$;

(iv) $A = \sum_{\lambda \in \mathcal{D}} \lambda E_\lambda$;

(v) For $\lambda \in \mathcal{D}$,

$$E_\lambda = \prod_{\mu \in \mathcal{D} \setminus \{\lambda\}} \frac{A - \mu I}{\lambda - \mu}.$$

A triple product

Consider the first q-Dolan/Grady relation:

$$[A, [A, [A, A^*]_q]_{q^{-1}}] = (q^2 - q^{-2})^2 [A^*, A].$$

For $\lambda, \mu \in \mathcal{D}$ let us multiply each side on the left by E_λ and the right by E_μ.

Evaluating the result using $E_\lambda A = \lambda E_\lambda$ and $AE_\mu = \mu E_\mu$, we obtain

$$\begin{aligned} & E_\lambda A^* E_\mu (\lambda - \mu)(q\lambda - q^{-1}\mu)(q^{-1}\lambda - q\mu) \\ & = E_\lambda A^* E_\mu (\mu - \lambda)(q^2 - q^{-2})^2. \end{aligned}$$

This yields

$$0 = E_\lambda A^* E_\mu (\lambda - \mu) P(\lambda, \mu),$$

where

$$P(\lambda, \mu) = \lambda^2 - (q^2 + q^{-2})\lambda\mu + \mu^2 + (q^2 - q^{-2})^2.$$

The polynomial P is **symmetric**:

$$P(\lambda, \mu) = P(\mu, \lambda).$$

From

$$0 = E_\lambda A^* E_\mu (\lambda - \mu) P(\lambda, \mu)$$

we get

$$E_\lambda A^* E_\mu = 0 \qquad \text{or} \qquad \lambda = \mu \qquad \text{or} \qquad P(\lambda, \mu) = 0.$$

Call λ, μ **adjacent** whenever $\lambda \neq \mu$ and $P(\lambda, \mu) = 0$.

The adjacency relation is **symmetric** since P is symmetric.

The graph $\mathcal{D}$

We just defined a symmetric binary relation on $\mathcal{D}$, called **adjacency**.

This relation turns $\mathcal{D}$ into an **undirected graph**.

The graph $\mathcal{D}$ is **connected**, because the O_q-module V is irreducible.

The graph $\mathcal{D}$, cont.

For the graph $\mathcal{D}$, each vertex is adjacent to **at most two** other vertices, because the polynomial P has degree two in each variable.

By these comments, the graph $\mathcal{D}$ is either a **path** or a **cycle**.

Shortly we will show that **a cycle cannot occur**.

The graph $\mathcal{D}$, cont.

Lemma

Referring to the graph $\mathcal{D}$, let λ denote a vertex that is adjacent to two distinct vertices μ, ν. Then

$$\mu - (q^2 + q^{-2})\lambda + \nu = 0.$$

Proof.

The vertices λ, μ are adjacent, so $P(\lambda, \mu) = 0$. The vertices λ, ν are adjacent, so $P(\lambda, \nu) = 0$. By these comments and the definition of P,

$$0 = \frac{P(\lambda, \mu) - P(\lambda, \nu)}{\mu - \nu} = \mu - (q^2 + q^{-2})\lambda + \nu.$$

$\square$

Lemma

We have $|\mathcal{D}| \geq 2$.

Proof.

By construction $|\mathcal{D}| \geq 1$. Suppose that $|\mathcal{D}| = 1$. Then A acts on V as a scalar multiple of the identity. The action of A^* on V is diagonalizable, so there exists a nonzero $v \in V$ that is an eigenvector for A^*. The subspace $\mathbb{F}v$ is invariant under A and A^*. Therefore $\mathbb{F}v = V$ since the O_q-module V is irreducible. Consequently $\dim V = 1$, for a contradiction. Therefore $|\mathcal{D}| \geq 2$. □

Ordering the vertices in the graph $\mathcal{D}$

Definition

Define $d = |\mathcal{D}| - 1$, and note that $d \geq 1$. Let $\{\theta_i\}_{i=0}^{d}$ denote an ordering of $\mathcal{D}$ such that θ_{i-1}, θ_i are adjacent for $1 \leq i \leq d$.

Lemma

The graph $\mathcal{D}$ is a path. Moreover, there exists $0 \neq a \in \mathbb{F}$ such that

$$\theta_i = aq^{d-2i} + a^{-1}q^{2i-d} \qquad (0 \le i \le d).$$

Proof: For notational convenience define $n = |\mathcal{D}|$. So $n = d + 1$. For the graph $\mathcal{D}$ let e denote the number of (undirected) edges. So $e = d$ if $\mathcal{D}$ is a path, and $e = n$ if $\mathcal{D}$ is a cycle. If $\mathcal{D}$ is a cycle then define $\theta_n = \theta_0$. For the graph $\mathcal{D}$ the vertices θ_{i-1}, θ_i are adjacent for $1 \le i \le e$. This yields

$$\theta_{i-1} - (q^2 + q^{-2})\theta_i + \theta_{i+1} = 0 \qquad (1 \le i \le e - 1).$$

The graph $\mathcal{D}$ is a path, cont.

(Proof continued..) For this recurrence the general solution is

$$\theta_i = aq^{d-2i} + \alpha q^{2i-d} \qquad\qquad (0 \leq i \leq e),$$

where $a, \alpha \in \mathbb{F}$. Using this general solution,

$$0 = P(\theta_0, \theta_1) = (q^2 - q^{-2})^2(1 - a\alpha).$$

The scalar $q^2 - q^{-2}$ is nonzero, so $a\alpha = 1$. Therefore $a \neq 0$ and $\alpha = a^{-1}$. It remains to show that the graph $\mathcal{D}$ is not a cycle. Assume that $\mathcal{D}$ is a cycle. By construction $e = n$ and $\theta_n = \theta_0$. Also by construction $n \geq 3$, so $d = n - 1 \geq 2$. This yields

$$0 = \frac{\theta_n - \theta_0}{\theta_d - \theta_1} = \frac{q^{2n} - 1}{q^{2d} - q^2}.$$

Therefore $q^{2n} = 1$, contradicting our assumption that q is not a root of unity. Consequently $\mathcal{D}$ is not a cycle, and the result follows. $\square$

The eigenspaces for A on V

For notational convenience, abbreviate $E_i = E_{\theta_i}$ for $0 \le i \le d$. By the construction,

$$E_i A^* E_j = 0 \quad \text{if} \quad |i-j| > 1 \qquad (0 \le i, j \le d).$$

For $0 \le i \le d$ let V_i denote the θ_i-eigenspace for A on V.

So $V_i = E_i V$.

How A^* acts on the eigenspaces for A

Lemma

We have

$$A^* V_i \subseteq V_{i-1} + V_i + V_{i+1} \qquad\qquad (0 \leq i \leq d),$$

where $V_{-1} = 0$ and $V_{d+1} = 0$.

Proof.

We have

$$A^* E_i = I A^* E_i = \sum_{\ell=0}^{d} E_\ell A^* E_i.$$

For $0 \leq \ell \leq d$, $E_\ell A^* E_i = 0$ if $|\ell - i| > 1$. The result follows. $\square$

How A acts on the eigenspaces for A^*

The previous lemma shows how A^* acts on the eigenspaces of A.

Interchanging the roles of A, A^* we see that A acts on the eigenspaces of A^* in the following way.

Lemma

There exists an ordering $\{V_i^*\}_{i=0}^{\delta}$ of the eigenspaces for A^* on V such that

$$AV_i^* \subseteq V_{i-1}^* + V_i^* + V_{i+1}^* \qquad (0 \leq i \leq \delta),$$

where $V_{-1}^* = 0$ and $V_{\delta+1}^* = 0$.

The eigenvalues for A^* on V

For $0 \leq i \leq \delta$ let θ^*_i denote the eigenvalue of A^* for the eigenspace V^*_i.

Lemma

With the above notation, there exists $0 \neq b \in \mathbb{F}$ such that

$$\theta^*_i = bq^{\delta-2i} + b^{-1}q^{2i-\delta} \qquad (0 \leq i \leq \delta).$$

Proof.

Interchange the roles of A and A^* in our earlier remarks. □

From our comments so far, we get the following result.

Theorem

The O_q-generators A, A^* act on the O_q-module V as a tridiagonal pair.

Definition

The tridiagonal pair in the previous theorem is said to have q-**Onsager type**.

Our next steps apply to every tridiagonal pair; not just those of q-Onsager type.

Until further notice, let A, A^* denote any tridiagonal pair on V.

STEP II: Show that $d = \delta$. This equation asserts that A and A^* have the same number of eigenspaces.

The concept of a tridiagonal system

To accomplish Step II, it is convenient to bring in the concept of a **tridiagonal system**.

This concept will be defined shortly.

Standard orderings of the eigenspaces for A

Consider our tridiagonal pair A, A^* on V.

An ordering $\{V_i\}_{i=0}^d$ of the eigenspaces of A is called **standard** whenever it satisfies

$$A^* V_i \subseteq V_{i-1} + V_i + V_{i+1} \qquad (0 \leq i \leq d).$$

If the ordering $\{V_i\}_{i=0}^d$ is standard then the inverted ordering $\{V_{d-i}\}_{i=0}^d$ is also standard, and no further ordering is standard.

Standard orderings of the primitive idempotents for A

An ordering of the primitive idempotents of A is called **standard** whenever the corresponding ordering of the eigenspaces of A is standard.

Similar comments apply to A^*.

Definition (Ito+Tanabe+Ter, 2001)

A **tridiagonal system on** V is a sequence

$$\Phi = (A; \{E_i\}_{i=0}^{d}; A^*; \{E_i^*\}_{i=0}^{\delta})$$

such that:

(i) A, A^* is a tridiagonal pair on V;

(ii) $\{E_i\}_{i=0}^{d}$ is a standard ordering of the primitive idempotents of A;

(iii) $\{E_i^*\}_{i=0}^{\delta}$ is a standard ordering of the primitive idempotents of A^*.

The relatives of a tridiagonal system

Until further notice, fix a tridiagonal system on V:

$$\Phi = (A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^\delta).$$

Note that each of the following is a tridiagonal system on V:

$$\begin{aligned}
\Phi^* &= (A^*; \{E_i^*\}_{i=0}^\delta; A; \{E_i\}_{i=0}^d), \\
\Phi^\downarrow &= (A; \{E_i\}_{i=0}^d; A^*; \{E_{\delta-i}^*\}_{i=0}^\delta), \\
\Phi^\Downarrow &= (A; \{E_{d-i}\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^\delta).
\end{aligned}$$

These tridiagonal systems are called **relatives** of Φ.

Definition

Recall our tridiagonal system

$$\Phi = (A; \{E_i\}_{i=0}^{d}; A^*; \{E_i^*\}_{i=0}^{\delta}).$$

For $0 \le i \le d$ let θ_i denote the eigenvalue of A for E_i.

We call $\{\theta_i\}_{i=0}^{d}$ the **eigenvalue sequence** of Φ.

For $0 \le i \le \delta$ let θ_i^* denote the eigenvalue of A^* for E_i^*.

We call $\{\theta_i^*\}_{i=0}^{\delta}$ the **dual eigenvalue sequence** of Φ.

We can now readily show that $d = \delta$.

For $0 \leq i \leq \delta$ and $0 \leq j \leq d$, define

$$V_{i,j} = (E_0^* V + E_1^* V + \cdots + E_i^* V) \cap (E_0 V + E_1 V + \cdots + E_j V).$$

For example,

$$V_{0,d} = E_0^* V, \qquad\qquad V_{\delta,0} = E_0 V.$$

The parameter N

Define

$$N = \min\{i + j | V_{i,j} \neq 0\}.$$

We have $V_{0,d} \neq 0$, so

$$N \leq d.$$

We have $V_{\delta,0} \neq 0$, so

$$N \leq \delta.$$

For $0 \leq i \leq N$ abbreviate $U_i = V_{i,N-i}$, so that

$$U_i = (E_0^* V + \cdots + E_i^* V) \cap (E_0 V + \cdots + E_{N-i} V).$$

The actions of A, A^* on $\{U_i\}_{i=0}^N$

Lemma

We have

(i) $(A^* - \theta_i^* I)U_i \subseteq U_{i-1}$ *for* $1 \leq i \leq N$;

(ii) $(A^* - \theta_0^* I)U_0 = 0$;

(iii) $(A - \theta_{N-i} I)U_i \subseteq U_{i+1}$ *for* $0 \leq i \leq N-1$;

(iv) $(A - \theta_0 I)U_N = 0$.

Proof.

(i) Use

$$(A^* - \theta_i^* I)(E_0^* V + \cdots + E_i^* V) = E_0^* V + \cdots + E_{i-1}^* V,$$
$$(A^* - \theta_i^* I)(E_0 V + \cdots + E_{N-i} V) \subseteq E_0 V + \cdots + E_{N-i+1} V.$$

The proof of (ii)–(iv) are similar. □

Corollary

We have

(i) $A^* U_i \subseteq U_{i-1} + U_i$ *for* $1 \leq i \leq N$;

(ii) $A^* U_0 \subseteq U_0$;

(iii) $AU_i \subseteq U_i + U_{i+1}$ *for* $0 \leq i \leq N-1$;

(iv) $AU_N \subseteq U_N$.

The subspaces $\{U_i\}_{i=0}^N$ span V

Lemma

$V = \sum_{i=0}^N U_i$.

Proof.

Define $W = \sum_{i=0}^N U_i$.

By construction, $W \neq 0$.

By the previous corollary, $AW \subseteq W$ and $A^*W \subseteq W$.

Therefore $W = V$. $\square$

Showing $d = \delta$

Lemma (Ito+Tanabe+Ter, 2001)

$N = d = \delta$.

Proof.

Recall that $N \leq d$ and $N \leq \delta$. By construction $U_i \subseteq E_0V + \cdots + E_{N-i}V$ for $0 \leq i \leq N$. Consequently $\sum_{i=0}^{N} U_i \subseteq \sum_{i=0}^{N} E_iV$. Observe that

$$\sum_{i=0}^{d} E_iV = V = \sum_{i=0}^{N} U_i \subseteq \sum_{i=0}^{N} E_iV.$$

Therefore $N = d$. Similarly we obtain $N = \delta$. □

Definition

We just showed $d = \delta$; this common value is called the **diameter** of Φ.

In the next few results, we describe the subspaces $\{U_i\}_{i=0}^{d}$ in more detail.

Lemma (Ito+Tanabe+Ter, 2001)

The sum $V = \sum_{i=0}^d U_i$ *is direct.*

Describing the subspaces $\{U_i\}_{i=0}^d$, cont.

Lemma (Ito+Tanabe+Ter, 2001)

For $0 \leq i \leq d$,

(i) $U_0 + \cdots + U_i = E_0^* V + \cdots + E_i^* V$;

(ii) $U_i + \cdots + U_d = E_0 V + \cdots + E_{d-i} V$.

STEP III: Use a **tetrahedron diagram** to describe the tridiagonal system

$$\Phi = (A; \{E_i\}_{i=0}^{d}; A^*; \{E_i^*\}_{i=0}^{d}).$$

A decomposition of V

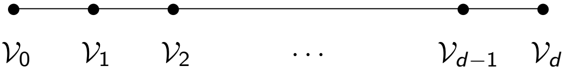
By a **decomposition of** V we mean a sequence $\{\mathcal{V}_i\}_{i=0}^{d}$ of nonzero subspaces whose direct sum is equal to V.

Let $\{\mathcal{V}_i\}_{i=0}^{d}$ denote a decomposition of V.

For $0 \leq i \leq d$ we call $\mathcal{V}_i$ the ith **component** of the decomposition.

Representing a decomposition of V by a diagram

Let $\{\mathcal{V}_i\}_{i=0}^d$ denote a decomposition of V. We describe this decomposition by the diagram



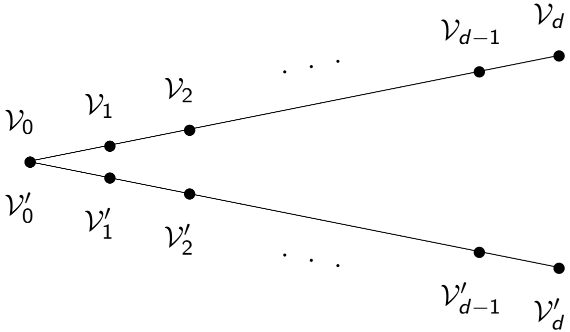
The labels $\mathcal{V}_i$ might be suppressed, if they are clear from the context.

Two related decompositions of V

Let $\{\mathcal{V}_i\}_{i=0}^d$ and $\{\mathcal{V}'_i\}_{i=0}^d$ denote decompositions of V. The condition

$$\mathcal{V}_0 + \mathcal{V}_1 + \cdots + \mathcal{V}_i = \mathcal{V}'_0 + \mathcal{V}'_1 + \cdots + \mathcal{V}'_i \qquad (0 \le i \le d)$$

will be described by the diagram



Illustrating the diagram convention

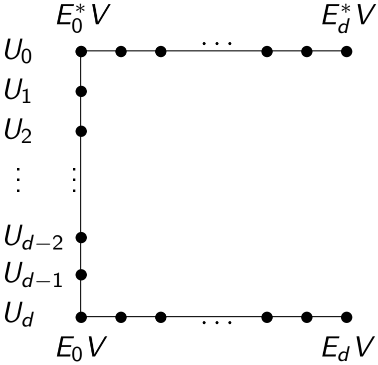
To illustrate the above diagram convention, consider the decomposition $\{U_i\}_{i=0}^d$ of V that we discussed earlier.

Recall that for $0 \leq i \leq d$,

$$U_0 + \cdots + U_i = E_0^* V + \cdots + E_i^* V,$$
$$U_i + \cdots + U_d = E_0 V + \cdots + E_{d-i} V.$$

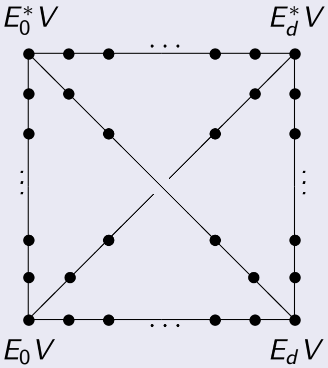
Illustrating the diagram convention, cont.

The corresponding diagram is shown below:



Theorem

We have

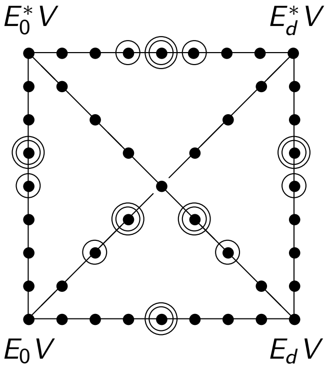


This theorem is proved by applying the diagram on the previous page to the relatives of Φ.

The diagram on the previous slide is called the **tetrahedron diagram** of Φ.

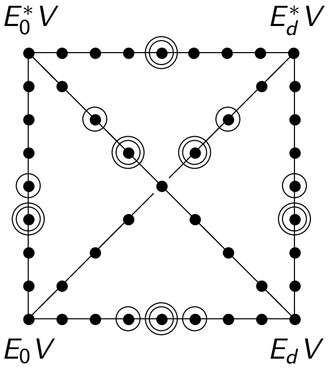
The following picture shows how A acts on the decompositions of V from the tetrahedron diagram, for $d = 8$:

A action:



The following picture shows how A^* acts on the decompositions of V from the tetrahedron diagram, for $d = 8$:

A^* action:



The shape of a decomposition

We now briefly consider the dimensions of the subspaces from the tetrahedron diagram.

Let $\{\mathcal{V}_i\}_{i=0}^d$ denote a decomposition of V.

By the **shape** of this decomposition, we mean the sequence

$$\{\dim \mathcal{V}_i\}_{i=0}^d.$$

Theorem (Ito+ Nomura + Tanabe+Ter, 2007)

The decompositions from the tetrahedron diagram all have the same shape.

Denoting this shape by $\{\rho_i\}_{i=0}^d$, we have

$$\begin{aligned}
\rho_i &= \rho_{d-i} & \qquad (0 \leq i \leq d), \\
\rho_{i-1} &\leq \rho_i & \qquad (1 \leq i \leq d/2), \\
\rho_0 &= 1.
\end{aligned}$$

We call $\{\rho_i\}_{i=0}^d$ the **shape of** Φ.

STEP IV: Classify the tridiagonal systems up to isomorphism.

We continue to discuss the tridiagonal system

$$\Phi = (A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d).$$

Recall the **eigenvalue sequence** $\{\theta_i\}_{i=0}^d$ of Φ and the **dual eigenvalue sequence** $\{\theta_i^*\}_{i=0}^d$ of Φ.

Earlier when we considered the special case of O_q-modules, we gave formulas for these eigenvalues.

In general, we can say the following.

The eigenvalue/dual eigenvalue sequence

Theorem (Ito + Tanabe + Ter, 2001)

For the tridiagonal system Φ, the scalars

$$\frac{\theta_{i-2}-\theta_{i+1}}{\theta_{i-1}-\theta_i}, \qquad \frac{\theta^*_{i-2}-\theta^*_{i+1}}{\theta^*_{i-1}-\theta^*_i}$$

are equal and independent of i for $2 \leq i \leq d-1$.

We bring in some notation.

Let x denote an indeterminate.

For $0 \leq i \leq d$ we define the polynomials

$$\begin{aligned}
\tau_i(x) &= (x-\theta_0)(x-\theta_1)\cdots(x-\theta_{i-1}),\\
\eta_i(x) &= (x-\theta_d)(x-\theta_{d-1})\cdots(x-\theta_{d-i+1}),\\
\tau^*_i(x) &= (x-\theta^*_0)(x-\theta^*_1)\cdots(x-\theta^*_{i-1}),\\
\eta^*_i(x) &= (x-\theta^*_d)(x-\theta^*_{d-1})\cdots(x-\theta^*_{d-i+1}).
\end{aligned}$$

A decomposition of V

We return our attention to the tetrahedron diagram of Φ.

One decomposition from the diagram is $\{\mathcal{U}_i\}_{i=0}^d$, where

$$\mathcal{U}_i = (E_0^* V + \cdots + E_i^* V) \cap (E_i V + \cdots + E_d V)$$

for $0 \leq i \leq d$. Note that

$$\mathcal{U}_0 = E_0^* V, \qquad \mathcal{U}_d = E_d V.$$

By our earlier comments,

$$
\begin{aligned}
&(A-\theta_i I)\mathcal{U}_i \subseteq \mathcal{U}_{i+1} \qquad (0 \leq i \leq d-1),\\
&(A-\theta_d I)\mathcal{U}_d = 0,\\
&(A^*-\theta^*_i I)\mathcal{U}_i \subseteq \mathcal{U}_{i-1} \qquad (1 \leq i \leq d),\\
&(A^*-\theta^*_0 I)\mathcal{U}_0 = 0.
\end{aligned}
$$

Definition (Ito + Tanabe + Ter, 2001)

There exists a unique $R \in \mathrm{End}(V)$ such that

$$R = A - \theta_i I \quad \text{on } \mathcal{U}_i \qquad (0 \le i \le d).$$

Note that

$$R\mathcal{U}_i \subseteq \mathcal{U}_{i+1} \qquad (0 \le i \le d-1), \qquad R\mathcal{U}_d = 0.$$

We call R the **raising map** for Φ.

The lowering map L

Definition (Ito + Tanabe + Ter, 2001)

There exists a unique $L \in \mathrm{End}(V)$ such that

$$L = A^* - \theta_i^* I \quad \text{on } \mathcal{U}_i \qquad (0 \leq i \leq d).$$

Note that

$$L\mathcal{U}_i \subseteq \mathcal{U}_{i-1} \qquad (1 \leq i \leq d), \qquad L\mathcal{U}_0 = 0.$$

We call L the **lowering map** for Φ.

A result about R

Theorem (Ito + Tanabe + Ter, 2001)

For $0 \leq i \leq j \leq d$ the map

$$R^{j-i} : \mathcal{U}_i \to \mathcal{U}_j$$

is injective (if $i + j \leq d$), bijective (if $i + j = d$), and surjective (if $i + j \geq d$).

Theorem (Ito + Tanabe + Ter, 2001)

For $0 \leq i \leq j \leq d$ the map

$$L^{j-i} : \mathcal{U}_j \to \mathcal{U}_i$$

is injective (if $i + j \geq d$), bijective (if $i + j = d$), and surjective (if $i + j \leq d$).

The split sequence

Using the maps R, L we now define a sequence of scalars $\{\zeta_i\}_{i=0}^d$.

Definition

For $0 \leq i \leq d$ the subspace $\mathcal{U}_0$ is invariant under $L^i R^i$; let ζ_i denote the corresponding eigenvalue.

So on $\mathcal{U}_0$,

$$L^i R^i = \zeta_i I.$$

We call $\{\zeta_i\}_{i=0}^d$ the **split sequence** for Φ. Note that $\zeta_0 = 1$.

A formula for the split sequence

For $0 \leq i \leq d$ let us examine ζ_i.

By construction the following holds on $\mathcal{U}_0$:

$$\begin{aligned}&(A^* - \theta^*_1 I) \cdots (A^* - \theta^*_{i-1} I)(A^* - \theta^*_i I) \\ &\times (A - \theta_{i-1} I) \cdots (A - \theta_1 I)(A - \theta_0 I) = \zeta_i I.\end{aligned}$$

In this equation, multiply each side on the left and right by E^*_0. After simplifying, this yields

$$E^*_0 \tau_i(A) E^*_0 = \frac{\zeta_i E^*_0}{(\theta^*_0 - \theta^*_1)(\theta^*_0 - \theta^*_2) \cdots (\theta^*_0 - \theta^*_i)}.$$

Two inequalities involving the split sequence

Lemma

With the above notation,

$$\zeta_d \neq 0.$$

Proof.

The scalar ζ_d is the eigenvalue of $L^d R^d$ on $\mathcal{U}_0$.

The maps $R^d : \mathcal{U}_0 \to \mathcal{U}_d$ and $L^d : \mathcal{U}_d \to \mathcal{U}_0$ are bijections, so $L^d R^d$ is nonzero on $\mathcal{U}_0$. $\square$

Lemma

We have

$$\zeta_d^{\Downarrow} \neq 0,$$

where

$$E_0^* \eta_d(A) E_0^* = \frac{\zeta_d^{\Downarrow} E_0^*}{(\theta_0^* - \theta_1^*)(\theta_0^* - \theta_2^*) \cdots (\theta_0^* - \theta_d^*)}.$$

Proof.

Apply the previous lemma to the relative

$$\Phi^{\Downarrow} = (A; \{E_{d-i}\}_{i=0}^{d}; A^*; \{E_i^*\}_{i=0}^{d}).$$

□

Lemma

We have

$$\zeta_d^{\Downarrow} = \sum_{i=0}^{d} \eta_{d-i}(\theta_0)\eta^*_{d-i}(\theta^*_0)\zeta_i.$$

Proof.

Use

$$\eta_d(x) = \sum_{i=0}^{d} \eta_{d-i}(\theta_0)\tau_i(x).$$

□

Definition

We call the sequence

$$\left(\{\theta_i\}_{i=0}^d; \{\theta_i^*\}_{i=0}^d; \{\zeta_i\}_{i=0}^d\right)$$

the **parameter array** of Φ.

A classification of tridiagonal systems

We now classify up to isomorphism the tridiagonal systems over $\mathbb{F}$.

A classification of tridiagonal systems

Theorem (Ito+Nomura+Ter, 2011)

Let $(\{\theta_i\}_{i=0}^d; \{\theta_i^*\}_{i=0}^d; \{\zeta_i\}_{i=0}^d)$ (1) *denote a sequence of scalars in* $\mathbb{F}$. *Then there exists a tridiagonal system* Φ *over* $\mathbb{F}$ *with parameter array (1) if and only if:*

(i) $\theta_i \neq \theta_j$, $\theta_i^* \neq \theta_j^*$ *if* $i \neq j$ $(0 \leq i, j \leq d)$;

(ii) *the expressions* $\frac{\theta_{i-2}-\theta_{i+1}}{\theta_{i-1}-\theta_i}$, $\frac{\theta^*_{i-2}-\theta^*_{i+1}}{\theta^*_{i-1}-\theta^*_i}$ *are equal and independent of* i *for* $2 \leq i \leq d-1$;

(iii) $\zeta_0 = 1$, $\zeta_d \neq 0$, *and*

$$0 \neq \sum_{i=0}^{d} \eta_{d-i}(\theta_0)\eta^*_{d-i}(\theta^*_0)\zeta_i.$$

Suppose (i)–(iii) *hold. Then* Φ *is unique up to isomorphism of tridiagonal systems.*

STEP V: We clarify which tridiagonal systems come from an O_q-module.

Theorem

Given a tridiagonal system

$$\Phi = (A; \{E_i\}_{i=0}^{d}; A^*; \{E_i^*\}_{i=0}^{d})$$

over $\mathbb{F}$, *with eigenvalue sequence* $\{\theta_i\}_{i=0}^{d}$, *dual eigenvalue sequence* $\{\theta_i^*\}_{i=0}^{d}$, *and* $d \geq 1$. *Then the following are equivalent:*

(i) *the tridiagonal pair* A, A^* *has q-Onsager type;*

(ii) *there exist nonzero* $a, b \in \mathbb{F}$ *such that*

$$\theta_i = aq^{d-2i} + a^{-1}q^{2i-d} \qquad (0 \leq i \leq d),$$
$$\theta_i^* = bq^{d-2i} + b^{-1}q^{2i-d} \qquad (0 \leq i \leq d).$$

Proof.

By our calculations in Step I. □

Summary

In this talk, we described the finite-dimensional irreducible O_q-modules V that satisfy a mild assumption.

We showed that the O_q-generators act on V as a tridiagonal pair.

We described the tridiagonal pairs and the related tridiagonal systems, using the concept of a tetrahedron diagram.

We classified up to isomorphism the tridiagonal systems, and explained which ones come from an O_q-module.

THANK YOU FOR YOUR ATTENTION!

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