The Simplicity of Gravitational Wave Merger Waveforms

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IMAPP, HEP Department Seminar
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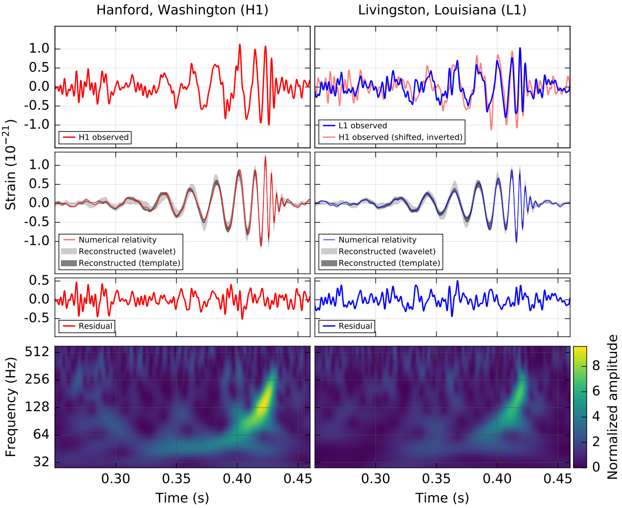
Outline

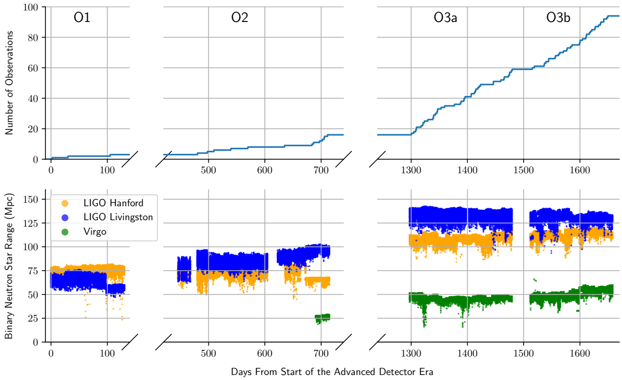
The signal from a BBH merger



Observed Events

- Following `Nitz+ 2021`: 94 events in total:
  - 90 Binary Black hole mergers,
  - 2 Binary Neutron Star Mergers
  - 2 Neutron Star – Black Hole mergers

Observed Events



Inspiral-Merger-Ringdown

The signal shows three distinct regimes:

- Inspiral: The two compact objects orbit each other, and the orbit decays due to GW emission – Increasing amplitude and frequency
- Merger: The objects coalesce and the remnant is formed – short burst of radiation
- Ringdown: The remnant object reaches equilibrium – Damped sinusoids or more complex post-merger for neutron stars

Inspiral-Merger-Ringdown

These three regimes allow us to probe different aspects of gravity.

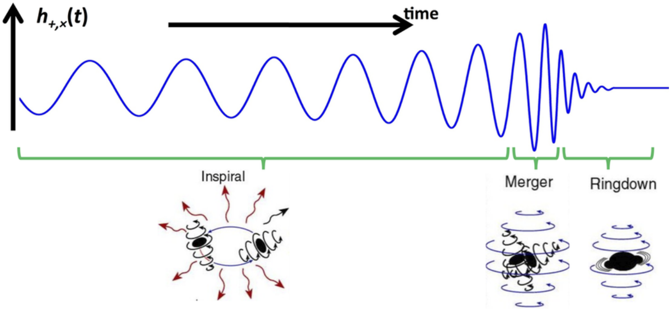
- Inspiral:
  - Tests of the post-Newtonian formalism
  - Tidal deformation of neutron stars and black holes
- Ringdown:
  - Black hole spectroscopy and black hole perturbations
  - dynamics and collapse of a hypermassive neutron star

Effects at the merger

- The merger is the least understood of the three different regimes
- The most promising probes here are "non-parametric" – i.e. no quantitative models, but rather general features
  - The black hole area increase law
  - The memory effect
  - Inspiral-Merger Consistency checks

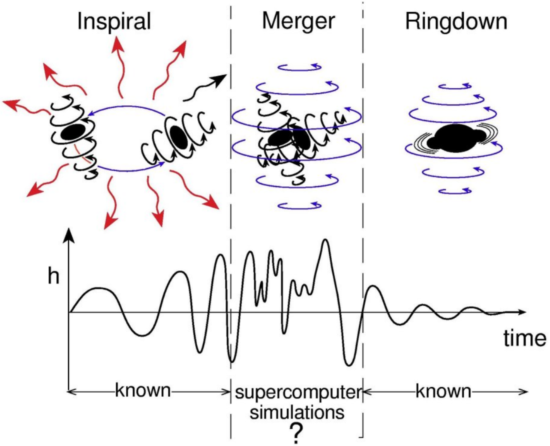
Why are merger waveforms so simple?

(joint work with J.L.Jaramillo)
We have “boring” or (“elegant”?) waveforms like these



Why are merger waveforms so simple?

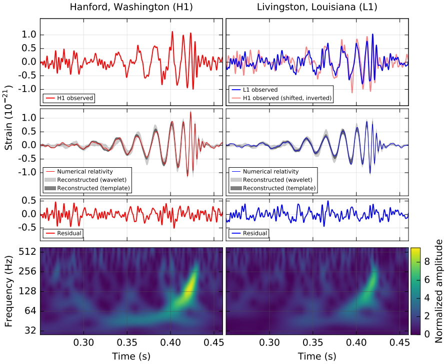
Why does this not happen?



(Attributed to Kip Thorne)

Merger waveform models

- Waveform models, including inspiral, merger and ringdown play a critical role in gravitational wave astronomy
- We need them for detecting and interpreting astrophysical signals – we will rarely be in the happy situation for GW150914 represented by these plots:



IMR waveform models

Even for GW150914 the waveform models were critically important for numerous reasons

- ...for establishing the detection beyond doubt, i.e. for calculating the statistical significance
- ...for calculating the parameters of the binary system
- ...for testing general relativity – do we really believe that we have modeled the waveforms with sufficient accuracy that we can challenge general relativity(?)

IMR Waveform models

Widely used waveform models

- Effective one-body
- Phenomenological
- Surrogate models

These models represent an important achievement now cover a large parameter space (moderate mass ratios, spins, precession...)

- believed to be sufficiently accurate for present generation detectors, but improvements required for third generation and LISA

Phenom, EOB and all that

- None of these are from "first principles", i.e. starting with the Einstein equations and making well controlled physically realistic assumptions
- All have significant input from Numerical relativity and Post-Newtonian theory
- Several have a key ansatz/guiding principle
  - EOB models binary system by an "equivalent" 1-body system moving in the background of a deformed Schwarzschild/Kerr metric
  - Phenom started with empirical observations of the amplitude and phase of Numerical relativity waveforms
- All of these have several phenomenological aspects, and for very loud signals, current models will not work – new input required

Why are merger waveforms so simple?

- Waveforms can be complicated, especially including precession and eccentricity
- The merger itself seems to be simple and universal and the details of the initial configuration do not matter– effacement, circularization, no hair theorem
- Better understanding should help us develop better waveform models
- For example, a better understanding of the inspiral-merger transition might improve our estimates of tidal deformability
- GR waveform tests of the merger might be put on the same footing as tests of the PN coefficients
- Might we expect similar simplicity in modified gravity theories?

The basic features of a BBH merger signal

There are two essential features:

- The waveform amplitude grows and "shuts off" immediately after the merger
- The signal transitions from an oscillatory regime to a damped regime
- Current approximation methods (PN or ringdown) diverge when extrapolated to the merger

These features point to a qualitative transition at the merger

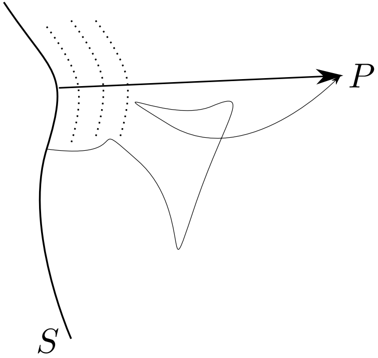
- Ringdown studies, close-limit approximation, and observed correlations in fluxes at BH and $\mathcal{I}^+$ suggest larger than expected validity of linear models

Geometric optics and Caustics

- A ray in geometrical optics defined by a wave vector $\mathbf{k} = (n\omega/c)\hat{\mathbf{k}}$ which can be written as the gradient of a scalar field
$$\mathbf{k} = (\omega/c)\nabla\Phi$$
- Several rays can pass through a given point, corresponding to several k and Φ – branches of a single multivalued function satisfying the Hamilton Jacobi equation
$$|\nabla\Phi|^2 = n^2$$

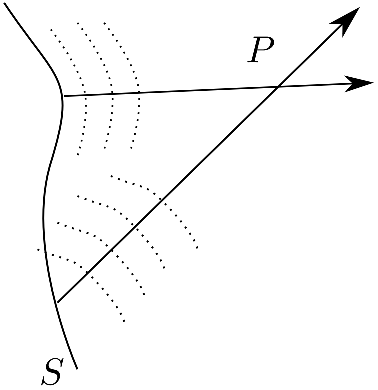
Geometric optics and Caustics

- $\Phi(s_1, s_2, \ldots; \mathbf{x})$ with "State variables" $(s_1, s_2, s_3, \ldots)$ can be solved by a variational principle



Geometric optics and Caustics

- In general we will have multiple rays reaching a point



Geometric optics and Caustics

- Considering rays in a homogeneous medium, we have two state variables representing the point on S

$$\Phi(s_1, s_2; \mathbf{x})$$

- For a given $\mathbf{x}$ we will generally have multiple solutions for the vanishing of the gradient

$$\frac{\partial \Phi}{\partial s_i} = 0$$

  This defines surface in $M \subset S \times X$
- Caustics occur when extrema merge and we have a singularity of the gradient map on M

$$\det \left[\frac{\partial^2 \Phi}{\partial s_i \partial s_j} \right] = 0$$

Caustics in Geometric optics

- The radiation field is given by the Fresnel/Fraunhofer integral

$$\psi(\mathbf{x}) = \left(\frac{k}{2\pi}\right)^{\frac{m}{2}} \int d^m\mathbf{s}\; e^{ik\Phi(\mathbf{s},\mathbf{x})} a(\mathbf{s},\mathbf{x})$$

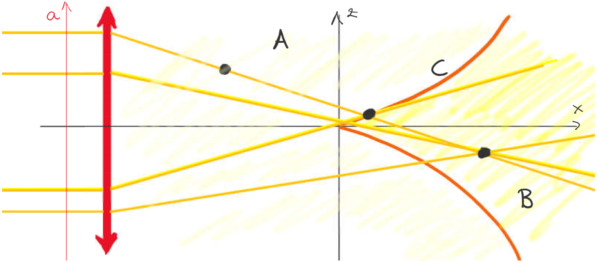
- Away from a caustic, the stationary phase approximation leads to

$$\psi(\mathbf{x}) \sim \sum_i \frac{e^{i(kS_i(\mathbf{x})+\alpha_i\pi/4)}}{|\det(\mathcal{M}^{-1}_{\alpha\beta}(\mathbf{x}))|^{\frac{1}{2}}} a(\mathbf{s}_i(\mathbf{x}),\mathbf{x})$$

- At the caustic this approximation fails and higher order expansions in k^{-1} do not help

Example 1: axisymmetric lens

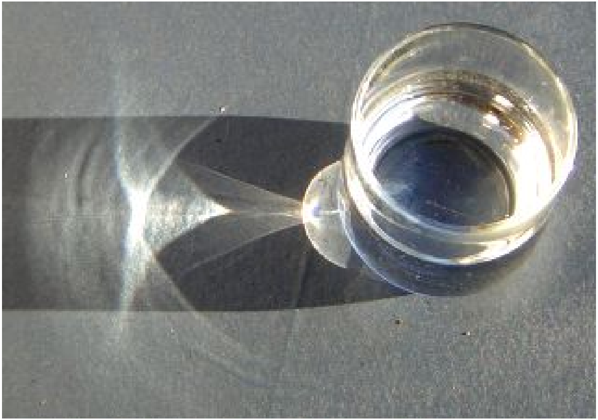
- Consider a lens with the focal length increasing radially outwards



Example 1: axisymmetric lens

- An ideal lens would have all rays passing through the focus
- Take the case where rays further away from the center see a larger focal length
- Region A: one ray passes through each point
- Region B: three rays pass through each point
- A and B are separated by a cusp C centered at the focus – This is the caustic
- The region B is more illuminated than A and a sharp transition occurs as we cross the caustic

Example 1: axisymmetric lens



Example 1: axisymmetric lens

- The different rays are parameterized by the impact parameter a
- Let $\Phi(a; x, y)$ be the phase of the electromagnetic field (the optical path length)
- The radiation field is given as

$$\psi \sim \int da\, e^{ik\Phi(a;x,z)} \qquad (1)$$

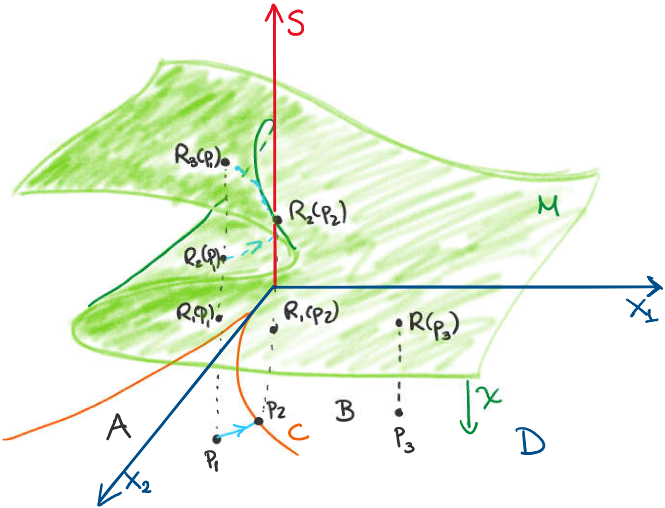
- The rays are determined by the condition

$$\partial_a \Phi(a; x, z) = 0$$

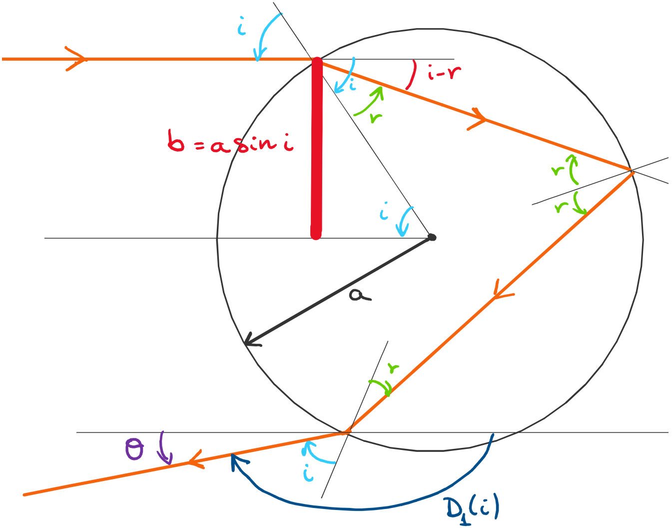
–This determines a surface in the space (a, x, z)

- The projection of the surface onto (x, z) determines the caustic
- For the lens we have a 4-order polynomial near the caustic, so that its derivative is cubic, and its projection yields a cusp

Example 1: axisymmetric lens



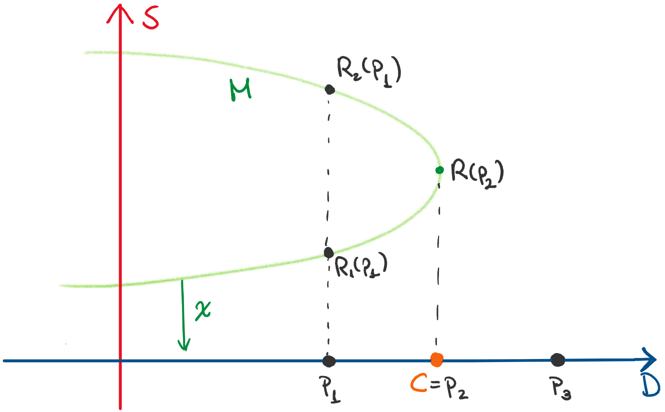
Example 2: rainbow



Example 2: rainbow

- As the impact parameter increases, the deflection angle decreases
- But beyond a critical value of $b = b_c$, the observer does not receive any rays with $D < D_c$ – This is the caustic
- Here the intensity is a maximum at the critical angle and then drops rapidly to zero
- Different wavelengths have different D_c which leads to a rainbow
- For $D > D_c$ two rays reach the observer, and none when $D < D_C$.
- Here $\Phi(b; D)$ turns out to be cubic in $b - b_c$, and thus its derivative is quadratic

Example 2: rainbow



Catastrophe theory

- These phenomena are well described in the framework of catastrophe theory
- There are universal forms of Φ near caustics, depending on the dimensions
- For both examples, we have 1 parameter characterizing the state of the incident ray (the impact parameter)
- For the lens, the observations are in a two dimensional space (x, z) and for the rainbow this is 1-dimensional (D)

Catastrophe theory

- For these dimensions, the universal functions are

$$\text{Rainbow}: \quad bx + b^3$$

$$\text{Lens}: \quad bx_1 + \frac{1}{2}b^2 x_2 + \frac{1}{4}b^4$$

  This is part of a classification due to R. Thom and V.I. Arnold
- Near the caustic, we can write the phase in these canonical forms by coordinate transformations and reparametrizations
- This restriction is based on the requirement of structural stability – preserved under small perturbations
- Structural stability is crucial for the observational importance of these phenomena – unstable cases do not occur in nature

BBH waveform as a fold catastrophe

- Hypothesis: The waveform $h(t)$ for the inspiral-merger transition is analogous to a caustic
- The space of observations ("control parameters") is just the time t
- Let us assume there is a single (so far abstract) state variable r
- It follows from the Arnold-Thom analysis that the phase must be diffeomorphism-equivalent to a cubic function

$$h(\tau(t)) \sim \int dr\, e^{i(\tau r + \frac{1}{3} r^3)}$$

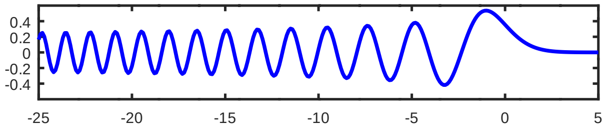
- τ is some function of t – we are allowed to make suitable reparametrizations to get the phase in this canonical form

BBH waveform as a fold catastrophe

- The above integral is of course just the Airy function

$$Ai(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dr\, e^{i(\tau r + \frac{r^3}{3})}$$

- The airy function is “boring”, just like a merger signal
- Satisfies this second order ODE: $u''(t) = tu(t)$
- $\rightarrow$ oscillatory for negative t and damped for positive t



BBH waveform as a fold catastrophe

- However, the Airy function by itself cannot be the answer – Amplitude grows, but frequency *decreases* as we go closer to the merger
- Under our hypothesis, the merger waveform must be, to leading order, a parameterized Airy function

$$h(t) \propto Ai(\tau(t))$$

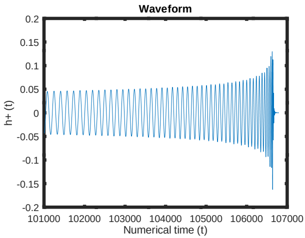
- The reparametrization is parameter dependent. So at leading order, it will depend on combinations of the two masses and it will reproduce the correct frequency evolution
- We are allowed to include a slowly varying amplitude
- For large negative x:

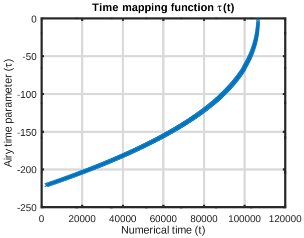
$$Ai(\tau) \sim \frac{1}{\sqrt{\pi}} \frac{1}{(-\tau)^{1/4}} \sin\left(\frac{2}{3}(-\tau)^{3/2} - \frac{\pi}{4}\right)$$

- This can easily be parameterized to reproduce the post-Newtonian phase and amplitude evolution

NR reparameterized to Airy

- As a simple exercise, we can reparameterize the pre-merger signal of a NR waveform to Airy – Use long $q = 7$ waveform from SXS catalog
- Align first peak of Airy function starting at $t = 0$ and moving towards negative t with peak of NR signal
- Mapping not well defined near $t = 0$ by this procedure





BBH waveform as a fold catastrophe

- After the merger, we get faster than exponential decay

$$Ai(\tau) \sim \frac{1}{\sqrt{\pi}} \frac{1}{(-\tau)^{1/4}} e^{-\frac{2}{3}\tau^{3/2}}$$

- The ringdown/post-merger is not part of this model and must be added separately
- Ringdown/post-merger is determined by the remnant object that is formed
- While we can parameterize τ, the rapid decay with no oscillations after the merger might lead to "accurate" fits with ringdown damping times – but this could in reality be a mis-identification of the ringdown phase
- For BNS mergers, inspiral-merger transition would be as described here, but the post-merger is of course much more complicated

New BBH merger model

Conventional model:

$$h_+(t) = A_+\eta(t)\cos(\varphi_0 + \varphi(t))\,,$$
$$h_\times(t) = A_\times\eta(t)\sin(\varphi_0 + \varphi(t))\,.$$

$\eta(t)$ is a slowly varying function of time and $A_{+,\times}$ depend on the inclination angle

$$A_+ = \frac{1+\cos^2\iota}{2}\,, \quad A_\times = \cos\iota$$

At the merger this parametrization breaks down

New BBH merger model – The post-Airy approximation

Our new proposal for the inspiral-merger transition

$$\begin{aligned} h_+(t;\varphi_0) &= A_+ a(t)\mathrm{Re}\left[Ai(\tau(t);\varphi_0)\right] , \\ h_\times(t;\varphi_0) &= A_\times a(t)\mathrm{Im}\left[Ai(\tau(t);\varphi_0)\right] . \end{aligned}$$

- $Ai(\tau(t);\varphi_0)$ is a phase shifted Airy function
- The amplitude $a(t)$ should now not diverge at the merger
- The re-parameterization $\tau(t)$ depends on the system parameters
- The ringdown needs to be added separately

Non-linear generalization of Airy

- Need a new ingredient near $t = 0$ – apart from ringdown, we look for non-linear generalization of Airy function
- New ingredient is the Painleve property
- For second order ODE

$$a(z)u'' + b(z)u'(z) + c(z) = 0$$

  Singularities of solution controlled by singularities of coefficients – integration constants do not lead to new singularities
- This is not true for non-linear equations
- Painleve property – no movable critical points (where general solution is multivalued)
- For example, we should not have solutions of the form $(z - c)^{1/2}$

Non-linear generalization of Airy

- The Painleve property can be broadly taken as a manifestation of effacement, circularization etc. – no strong dependence on initial conditions
- There are 6 Painleve equations (out of 50) which cannot be solved in terms of known special functions
- We propose Painleve-II as a suitable generalization of the Airy function

$$u'' = 2u^3 + tu + \alpha$$

- Near $t = 0$ the non-linear term dominates (ignoring α, local behavior is of the form $u = 1/(t + \beta)$)
- But away from the small t region, P_{II} is essentially indistinguishable from Airy

Conclusion and even more speculation

- This analysis is not (yet) meant to be an accurate IMR model – it is meant to be a leading order model for the merger
- There is a close connection of this analysis with well known WKB (or "turning point") problems
- GR will lead to corrections to this model, but unlike PN or ringdown, the deviations should be uniform (i.e. no infinities at the merger)
- We did not use GR anywhere in this argument but the construction is stable under perturbations – similar universality should hold thus for all theories with small deviations from GR
- The occurrence of the Painleve-II equation suggests integrability hidden in the Einstein equations for the BBH problem