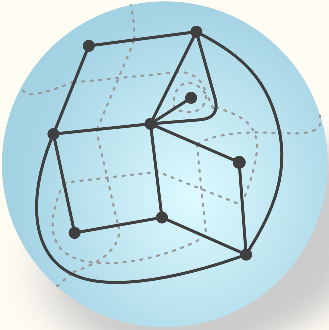
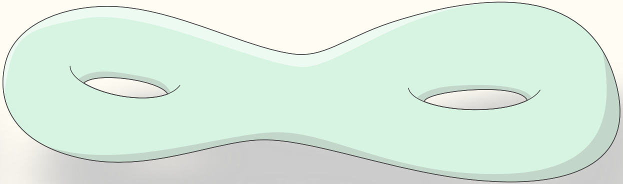
Gravity without Coordinates? - *Some Mathematics*

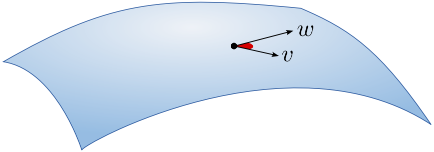
Timothy Budd





Metric vs geometry

- GR: gravitational degrees of freedom encoded in the (Riemannian/Lorentzian) metric $g_{\mu\nu}(x)$ on spacetime manifold



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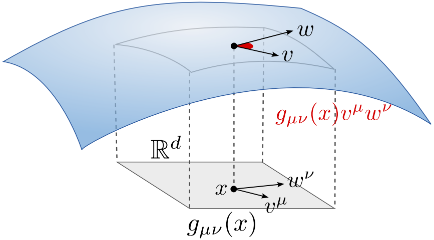

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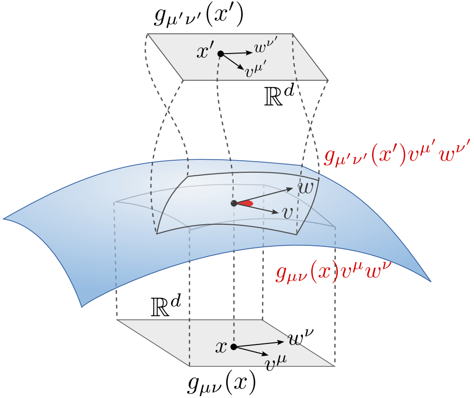

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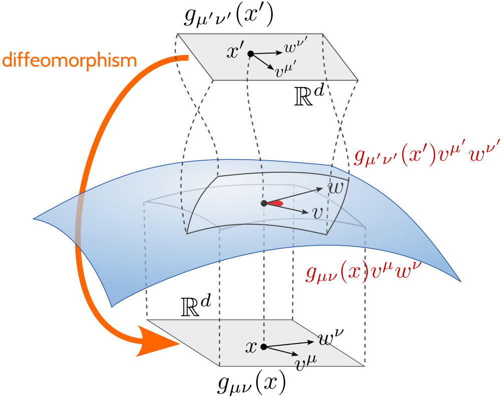

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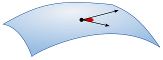
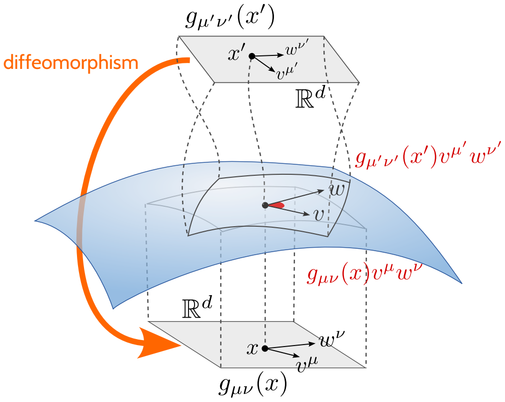
$$\mathcal{M} = \{\text{Metrics } g_{\mu\nu}\} \,/\, \mathrm{Diff} = \Bigg\{$$  $$\Bigg\}$$



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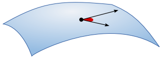
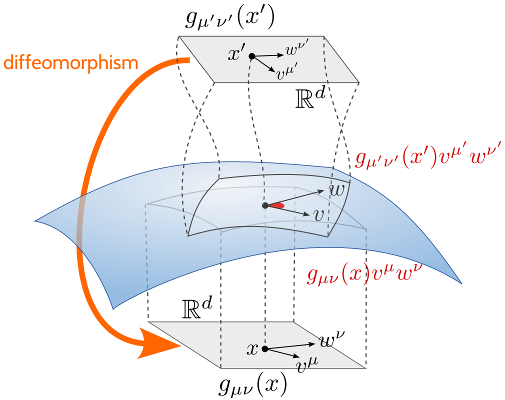


- In general, a very complicated non-linear ∞-dim mathematical space, but understanding restricted versions is important in many applications.



Moduli spaces: why?

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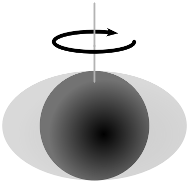
$$\mathcal{M} = \left\{ \begin{array}{c} \text{stationary asymptotically flat} \\ \text{black hole solutions to GR} \end{array} \right\}$$



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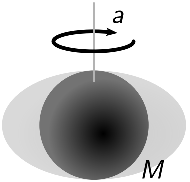
Mass (M)



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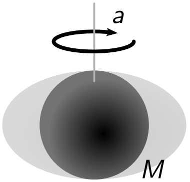

- ...or directly relevant to observations

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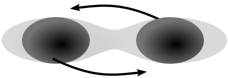

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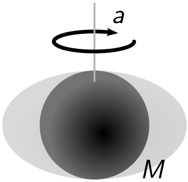

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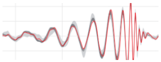

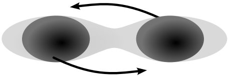

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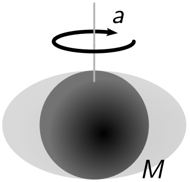

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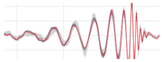

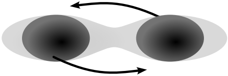

- and crucial to (path integral approaches to) quantum gravity

$$Z = \int \frac{D[g_{\mu\nu}]}{\mathrm{Diff}} e^{-S[g_{\mu\nu}]}$$

Diff: don't want to integrate over gauge

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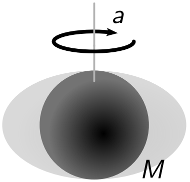

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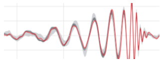
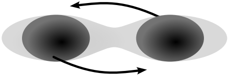
$$\mathcal{M} = \left\{ \begin{array}{c} \text{asymptotically flat binary} \\ \text{black hole solutions to GR} \\ \text{(plus position/orientation detector)} \end{array} \right\} \longrightarrow \left\{ \vphantom{\begin{array}{c} a \\ b \\ c \end{array}} \right.$$  $$\left. \vphantom{\begin{array}{c} a \\ b \\ c \end{array}} \right\}$$



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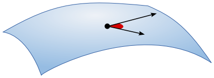

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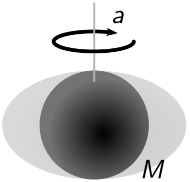

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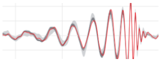

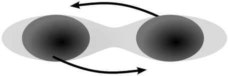

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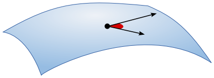

$\mathcal{M}$ requires a natural measure

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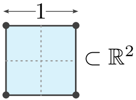

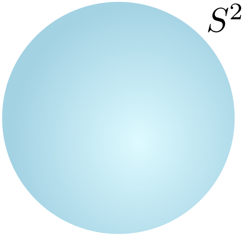

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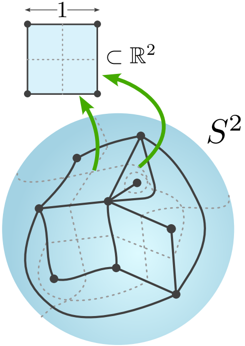

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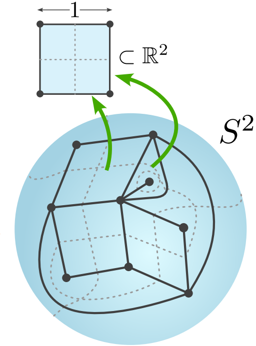

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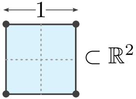

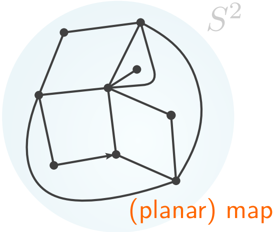

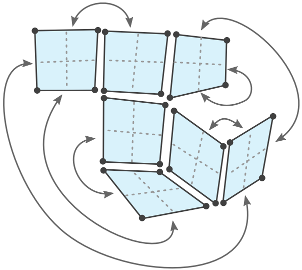

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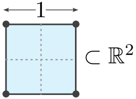

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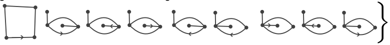
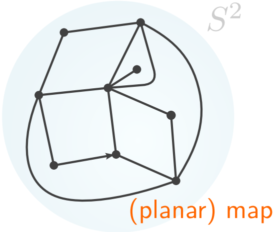
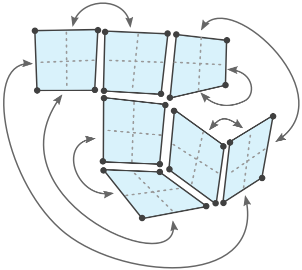
- This is a finite space, entirely determined by combinatorial data. $\mathcal{M}_2^\square = \{$  $\}$





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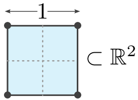

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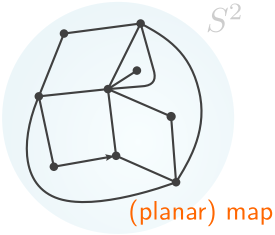
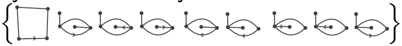


- This is a finite space, entirely determined by combinatorial data. $\mathcal{M}_2^{\square} = \{$  $\}$
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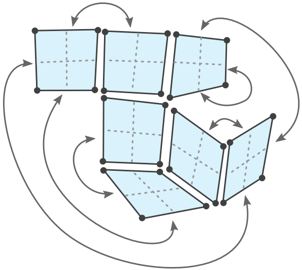

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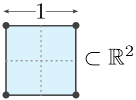

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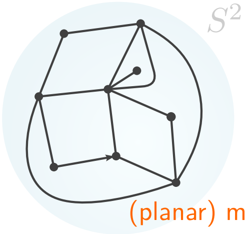
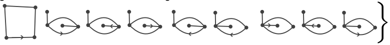


- This is a finite space, entirely determined by combinatorial data. $\mathcal{M}_2^{\square} = \{$  $\}$
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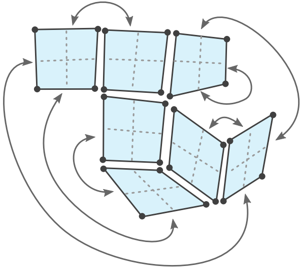
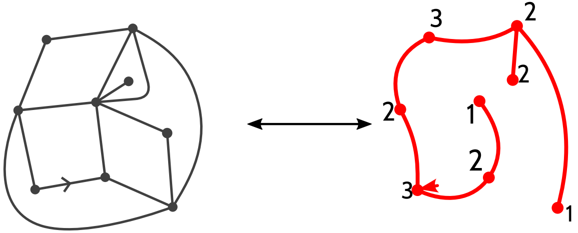


- But how to count? And how to determine statistical properties of the random geometry determined by Z?

The bijective approach

- There exists a bijection [Cori, Vauquelin] [Schaeffer, '99]

$$\mathcal{M}_n^{\square} \xleftrightarrow{(n+2)\text{-to-}2} \left\{ \begin{array}{c} \text{rooted plane trees with labels} \in \mathbb{N} \\ \text{that vary by at most 1} \end{array} \right\}$$



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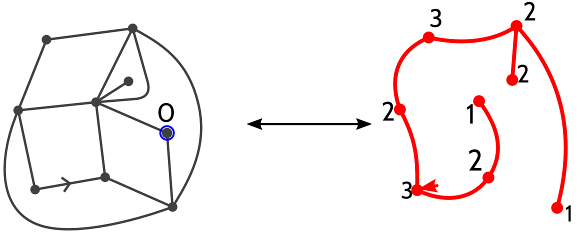

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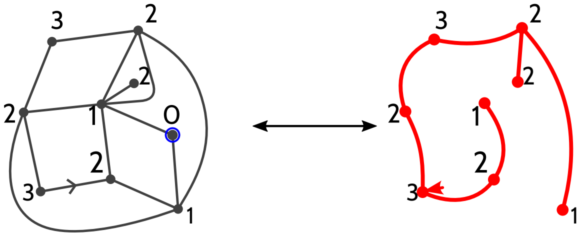

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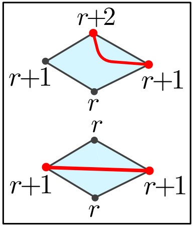

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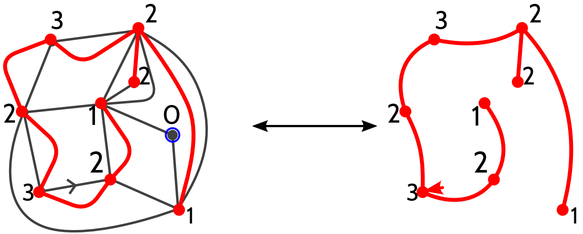

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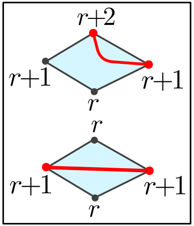

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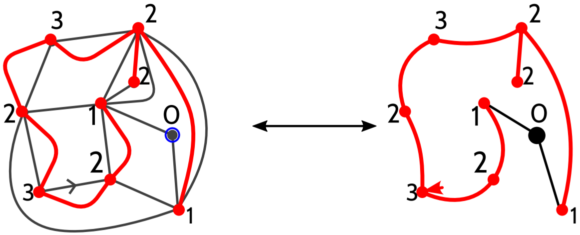

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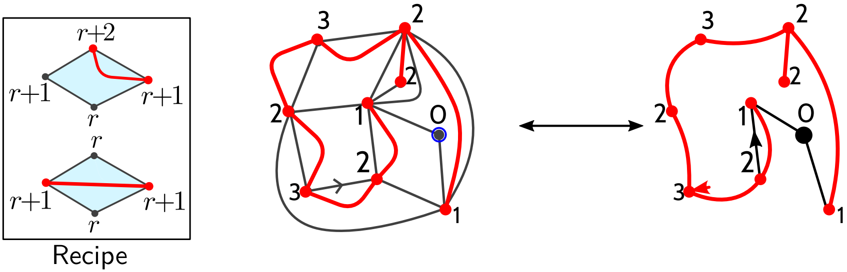

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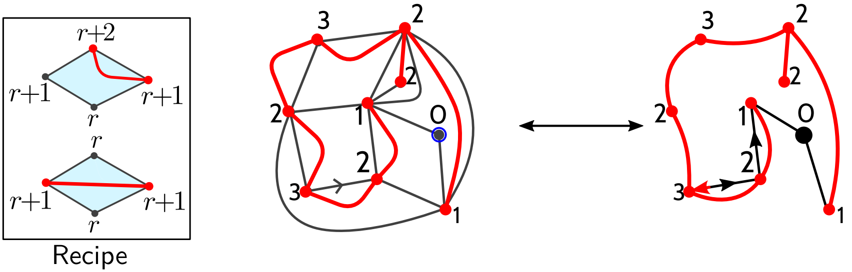

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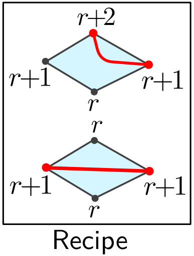

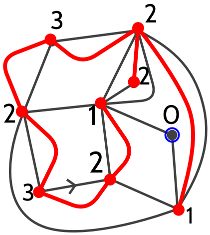

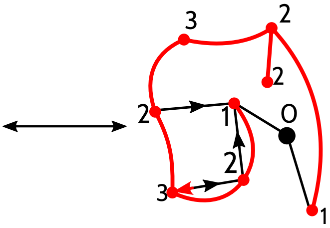

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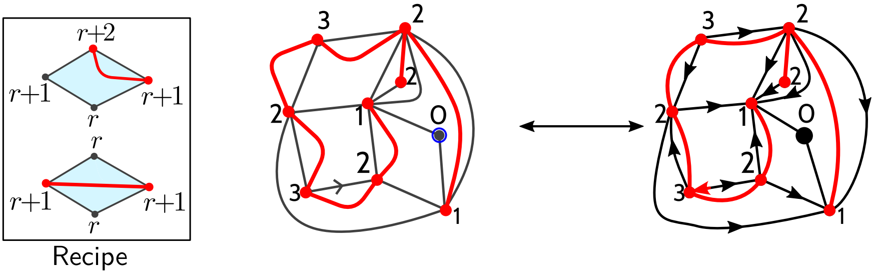

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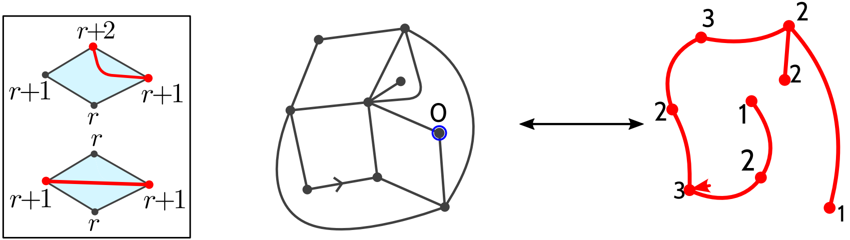

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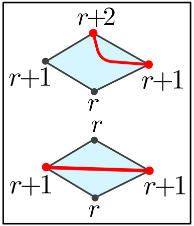
Labelings

Catalan numbers

Vertices



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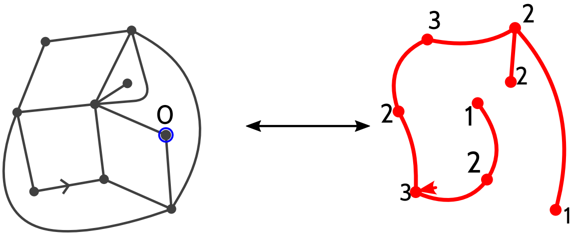

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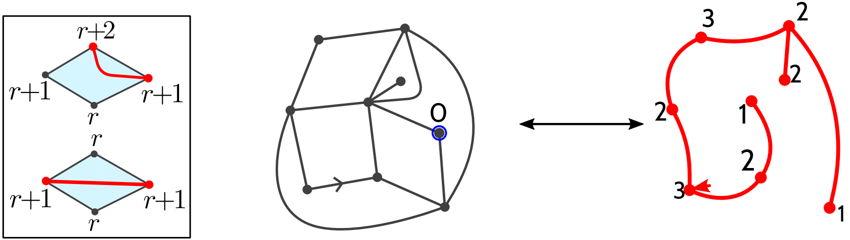
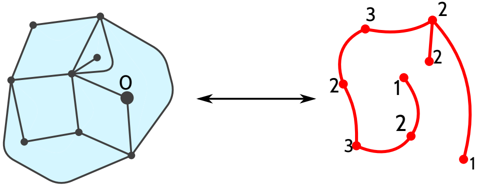
Labelings · Catalan numbers · Vertices

- Uniform measure on $\mathcal{M}_n^{\square}$ $\iff$ uniform plane tree + uniform labeling.



Recipe

The bijective approach: a detailed picture of the geometry



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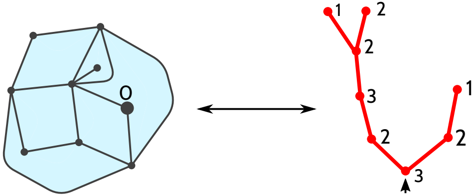

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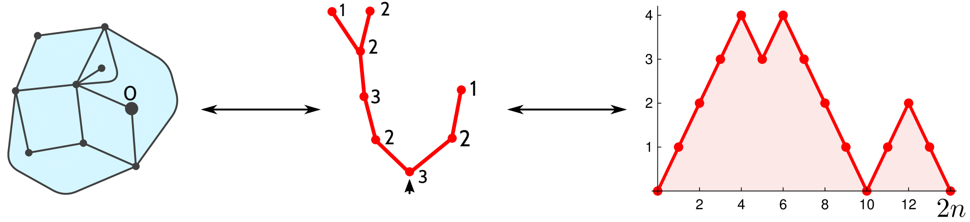

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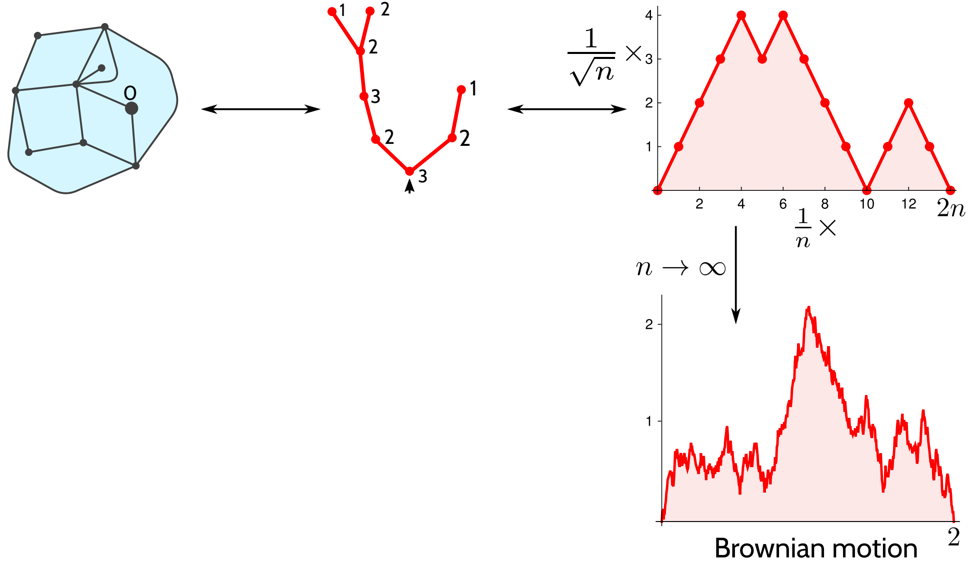

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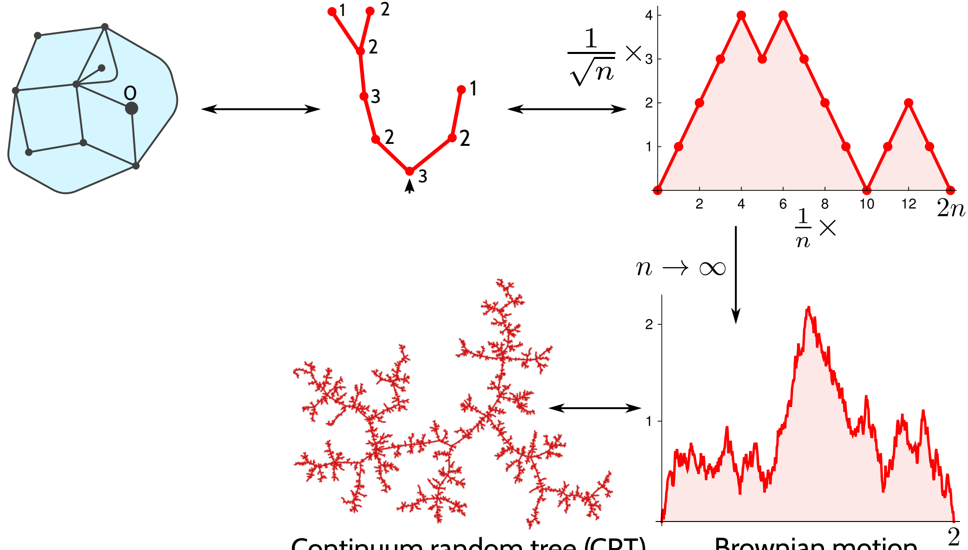
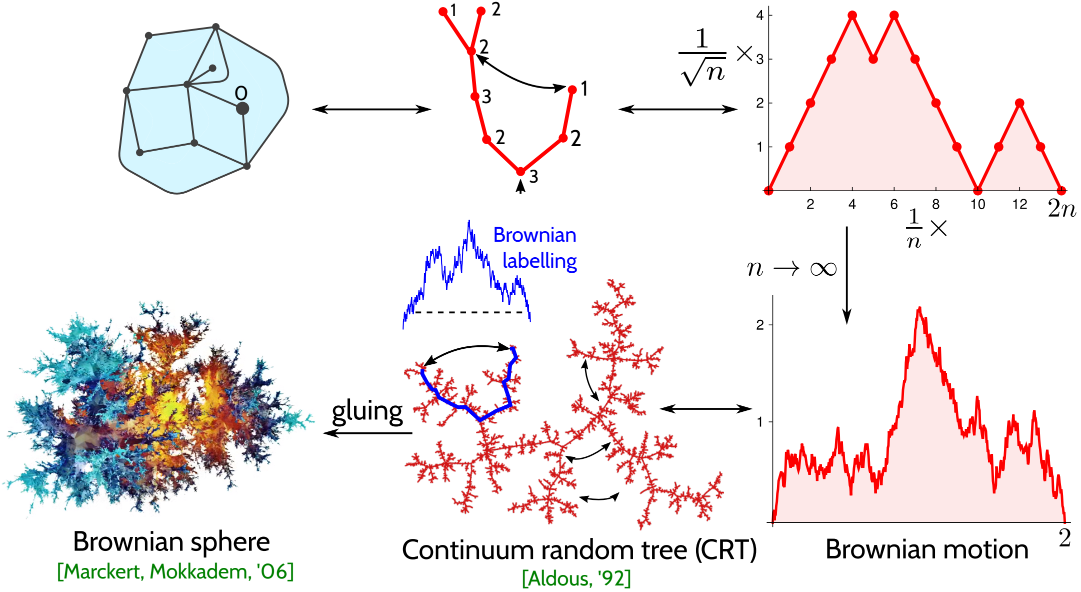


Continuum random tree (CRT)

[Aldous, '92]

Brownian motion

The bijective approach: a detailed picture of the geometry



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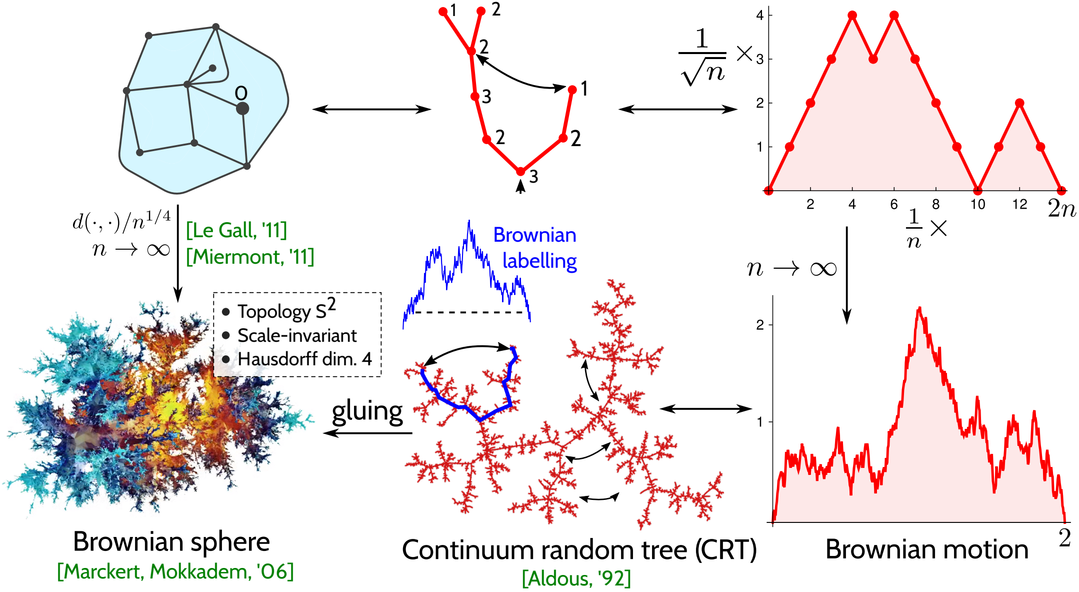
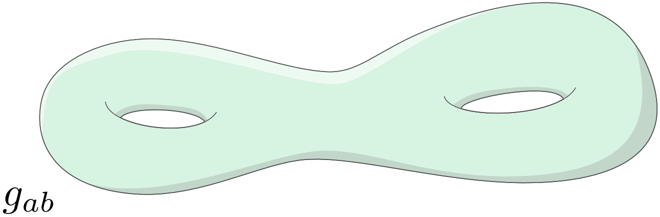


Hyperbolic geometry

- Constant curvature: Ricci scalar $R = -2 \iff$ Gaussian curvature $K = -1$.



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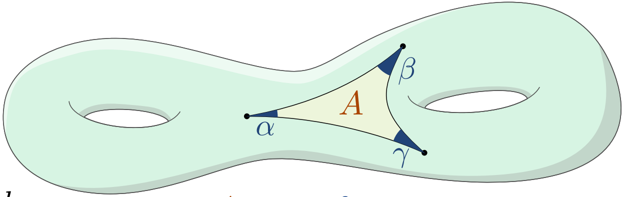

g_{ab}

$A + \alpha + \beta + \gamma = \pi$

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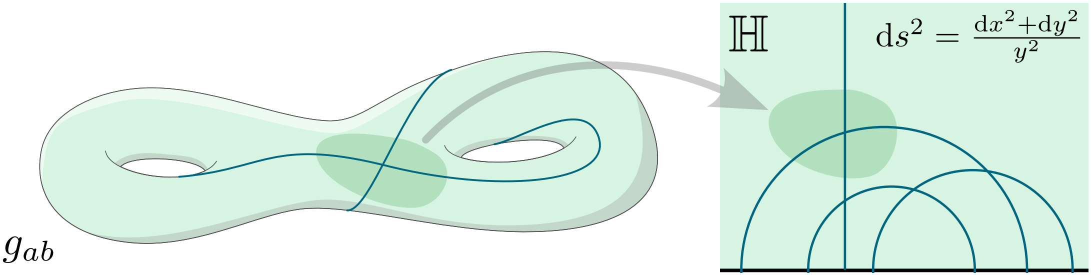

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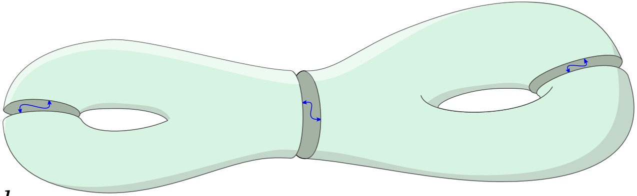

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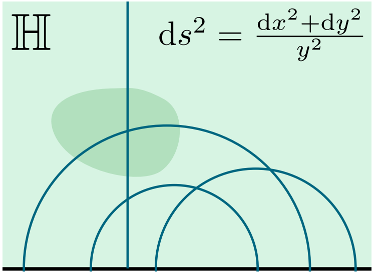
Pair of pants decomposition



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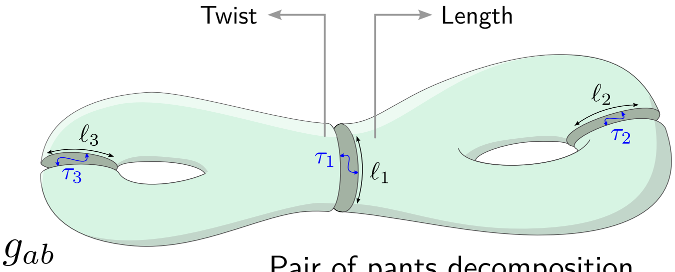
- Geometry is determined by Fenchel-Nielsen coordinates $\ell_1, \tau_1, \ell_2, \tau_2, \ldots$, but pants decomposition is far from unique.



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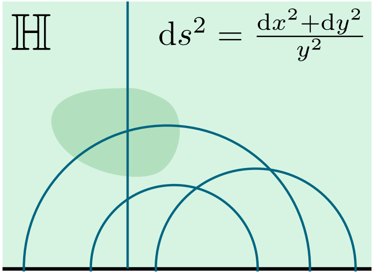

Moduli space of hyperbolic surfaces [Wolpert, Penner, Zograf, Witten, Kontsevich, Mirzakhani, ...]

- Consider the Moduli space

$$\mathcal{M}_{g,n}^{\mathrm{hyp}}(\mathsf{L}) = \left\{ \begin{array}{l} \text{hyperbolic metrics on genus-}g\text{ surface with } n \\ \text{geodesic boundaries of lengths } \mathsf{L} = (L_1, \ldots, L_n) \end{array} \right\} / \mathrm{Diff}$$

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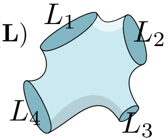

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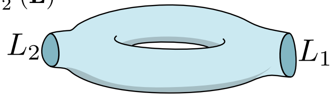

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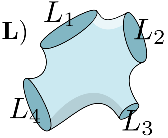

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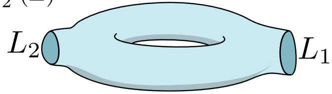

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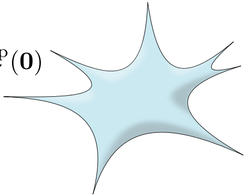

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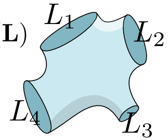

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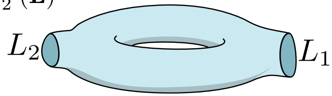

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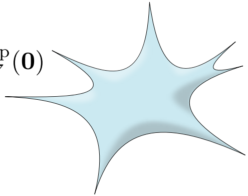

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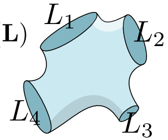

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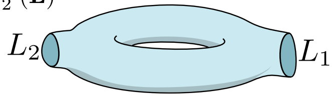

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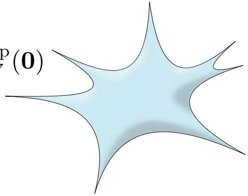

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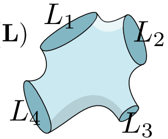
$\mathcal{M}_{0,4}^{\mathrm{hyp}}(\mathbf{L})$



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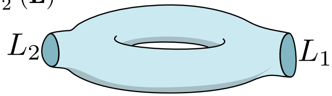

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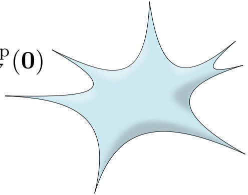
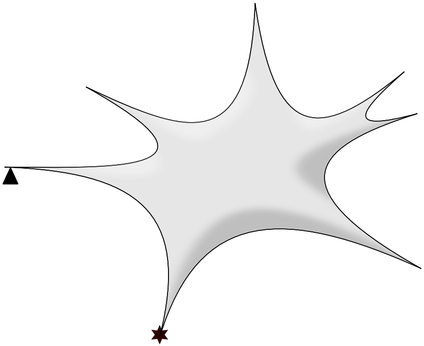


Bijective approach to hyperbolic surfaces?

- Where is the tree? Determine cut locus of a distinguished cusp.



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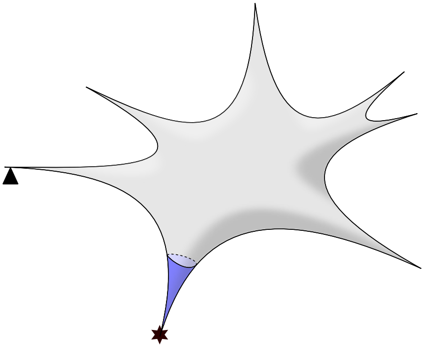

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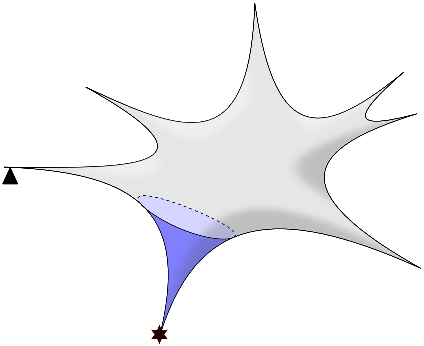

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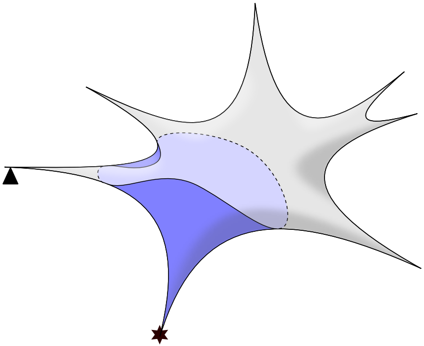

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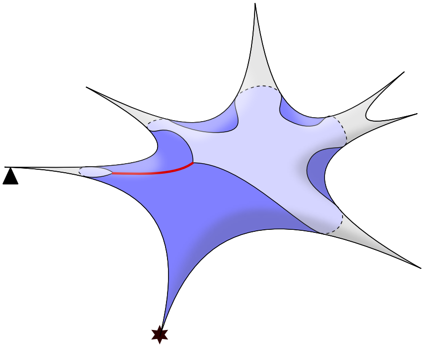

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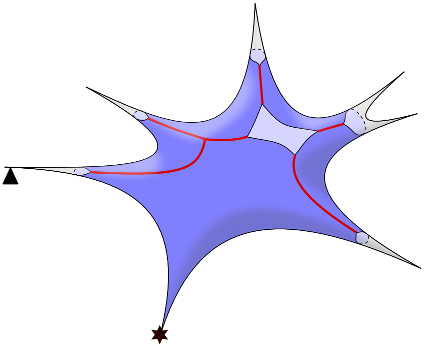

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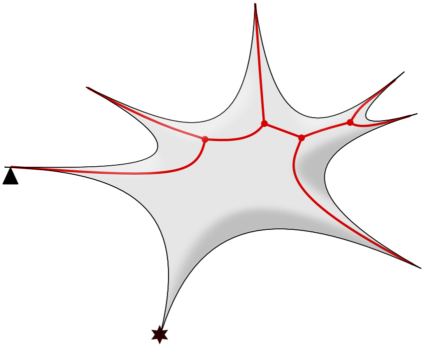

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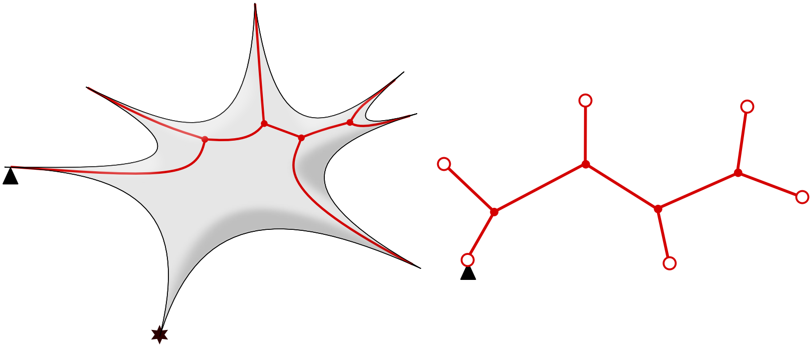

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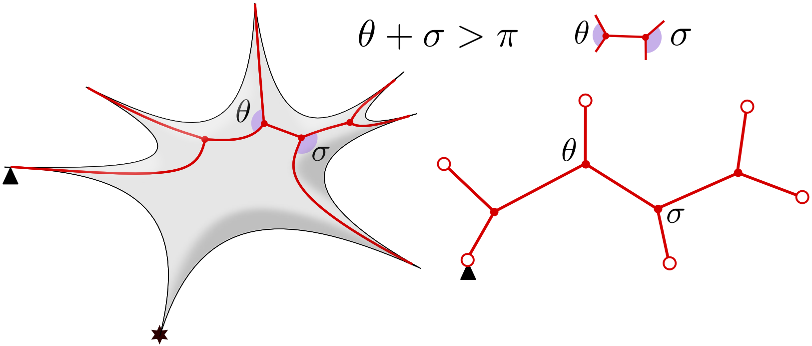

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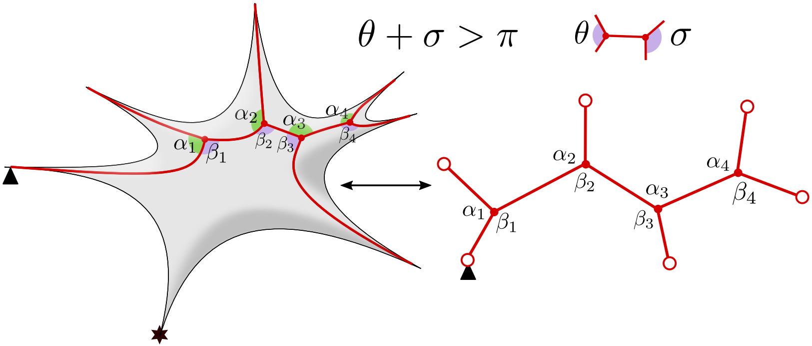

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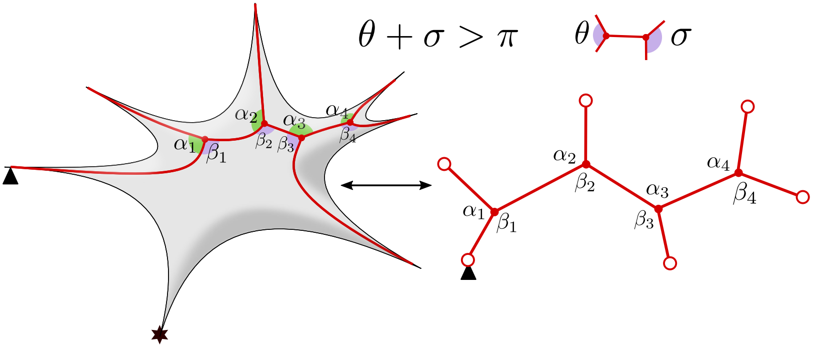
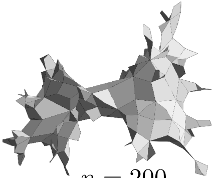
- $\mathrm{d}\mu_{\mathsf{WP}}$ is mapped to Euclidean measure $2^{n-3}\mathrm{d}\alpha_1\mathrm{d}\beta_1 \cdots \mathrm{d}\alpha_{n-3}\mathrm{d}\beta_{n-3}$.



Application: same scaling limit as planar maps [TB, Curien, '22+]

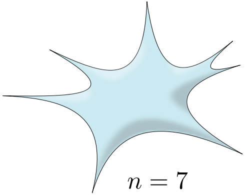
Nicolas Curien

Planar maps with n quadrangles



$n = 200$

Hyperbolic surfaces with n cusps



$n = 7$

Application: same scaling limit as planar maps [TB, Curien, '22+]

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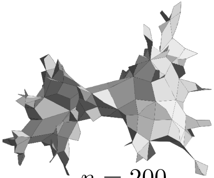

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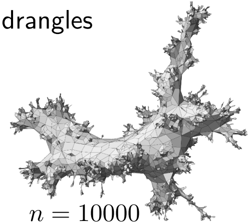
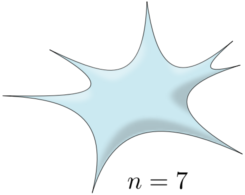


$n = 10000$

Hyperbolic surfaces with n cusps



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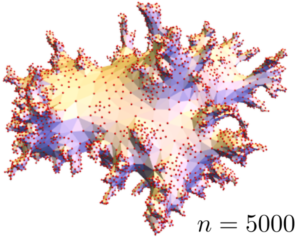

$n = 5000$

Application: same scaling limit as planar maps [TB, Curien, '22+]

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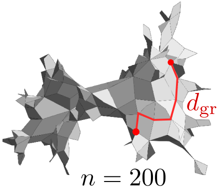
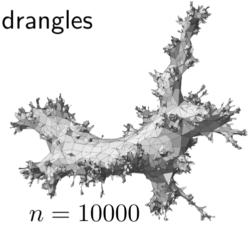
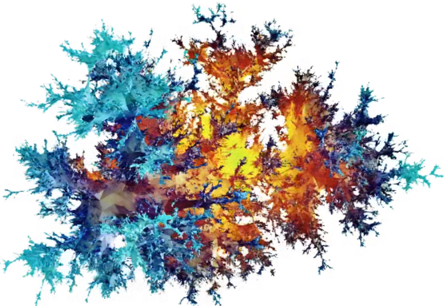




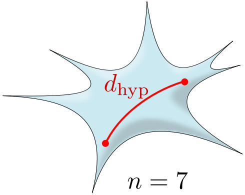
$\frac{1}{n^{1/4}} d_{\mathrm{gr}}$

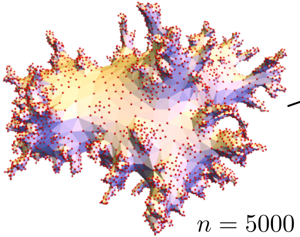
$n \to \infty$



Brownian geometry

Hyperbolic surfaces with n cusps





$n \to \infty$

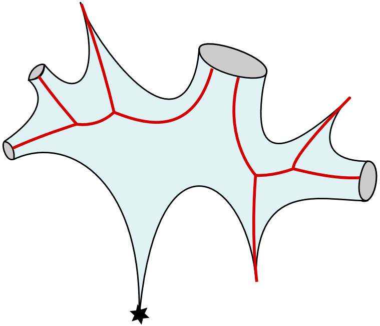
$\frac{1}{n^{1/4}} d_{\mathrm{hyp}}$

Extension to surfaces with geodesic boundaries


Thomas Meeusen
(master)


Bart Zonneveld

- How about $\mathcal{M}_{0,n}^{\text{hyp}}(\mathrm{L})$ with $\mathrm{L} \neq 0$? Does it admit a tree bijection?



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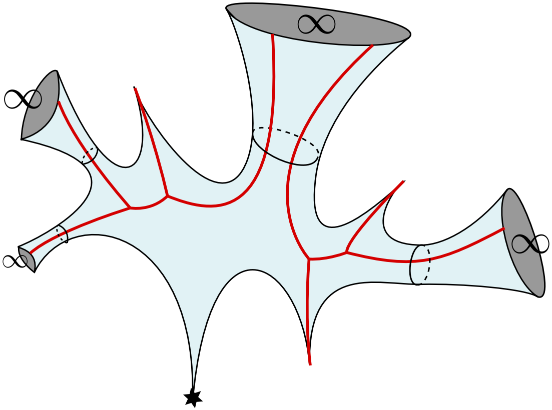

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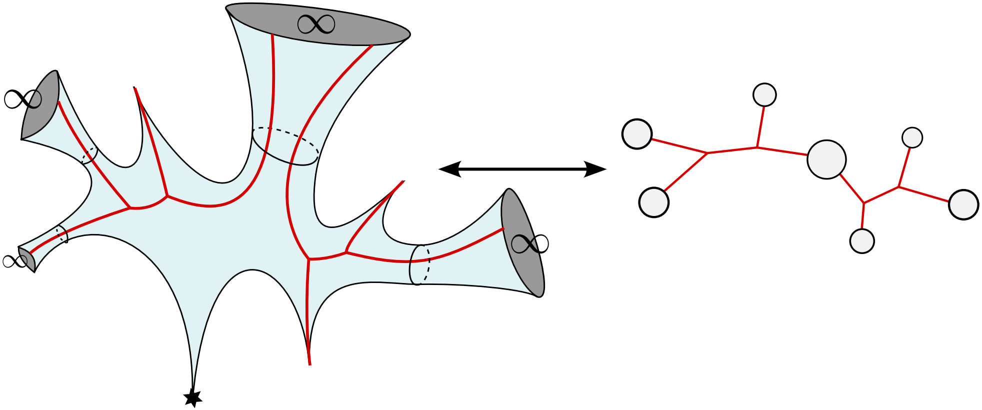

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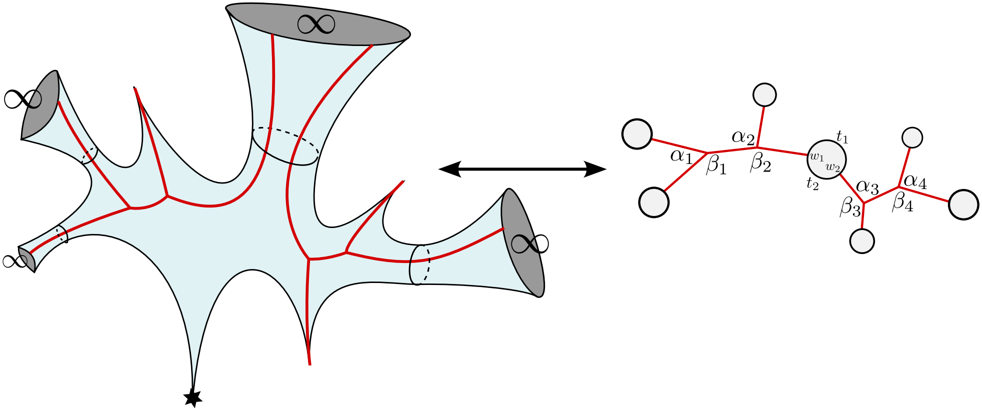

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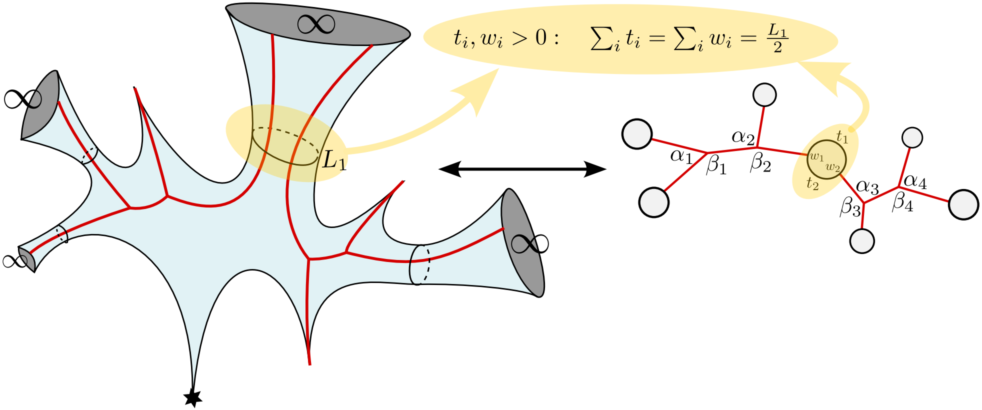
- Now $\mathrm{d}\mu_{\mathrm{WP}} \leftrightarrow 2^{n-3}\mathrm{d}\alpha_1\mathrm{d}\beta_1\cdots\mathrm{d}\alpha_k\mathrm{d}\beta_k\mathrm{d}t_1\mathrm{d}w_1\cdots\mathrm{d}t_m\mathrm{d}w_t$, and can reproduce $V_{0,n}(\mathrm{L})$.

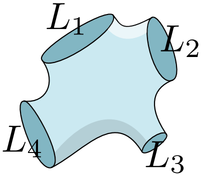


Bijection between hyperbolic surfaces and maps?

Alicia Castro

- A mysterious identity between WP volumes and certain maps: [TB, '20]

$$\mathrm{Vol}\left(\mathcal{M}_{g,n}^{\mathrm{hyp}}(\mathrm{L})\right) = \mathrm{Vol}\left(\left\{\begin{array}{l} 2\pi\text{-irreducible metric maps on genus-}g\text{ surface} \\ \text{with } n \text{ faces of circumference } \alpha_i = \sqrt{L_i + 4\pi^2} \end{array}\right\}\right) \quad \text{for } g = 0, 1.$$



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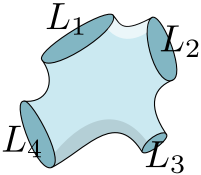

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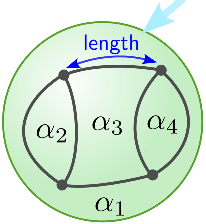

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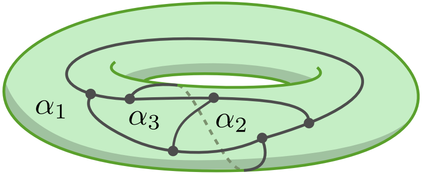

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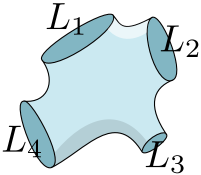

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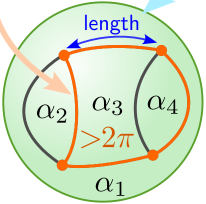

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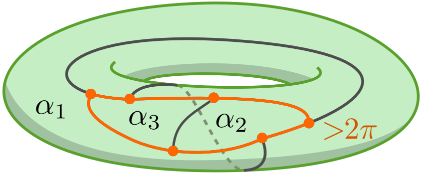

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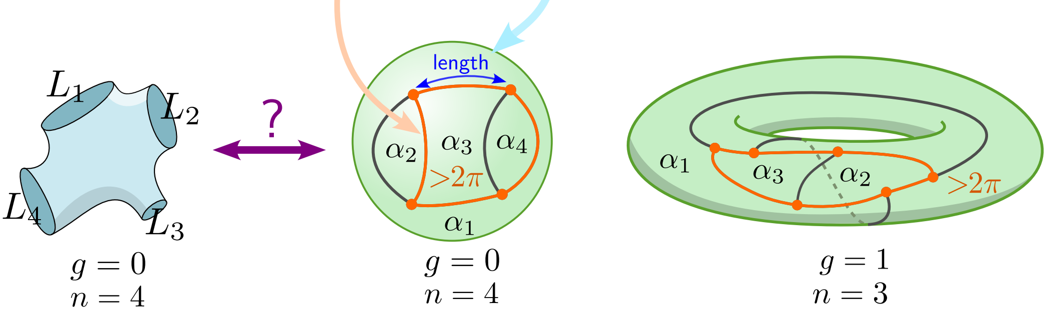
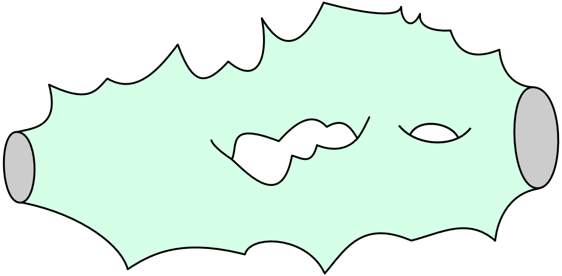


- Is there a bijective interpretation? Would shed light on a matrix model interpretation of JT gravity. [Saad, Shenker, Stanford, '19]

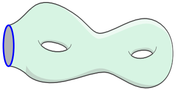
Higher genus and beyond constant curvature?

Bart Zonneveld

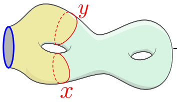
- Mirzakhani's topological recursion admits a generalization to surfaces with defects if boundaries are taken tight. [TB, Zonneveld, '22+]



$2\frac{\partial}{\partial L_1} L_1$



$= \int_0^\infty \mathrm{d}x \int_0^\infty \mathrm{d}y\, xy\, K_0(x+y, L_1)$



$+ \cdots$

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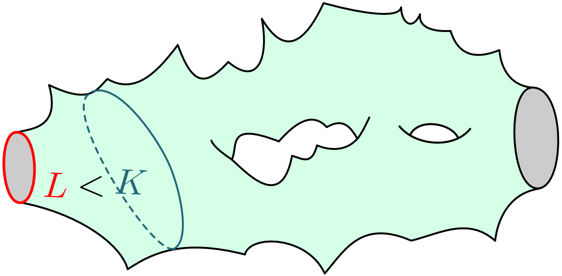
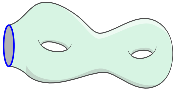
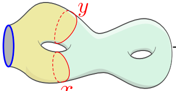


$2\frac{\partial}{\partial L_1} L_1$  $= \int_0^\infty \mathrm{d}x \int_0^\infty \mathrm{d}y \, xy \, K_0(x+y, L_1)$  $+ \cdots$

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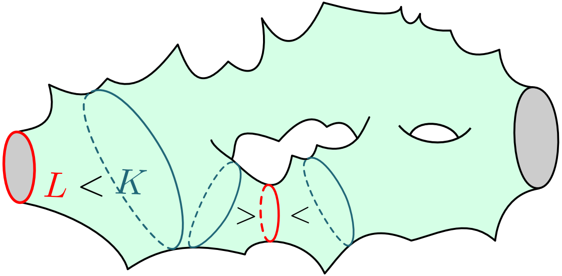

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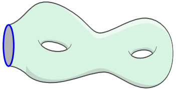
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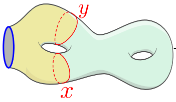

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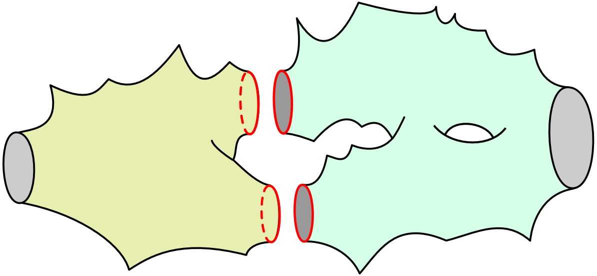
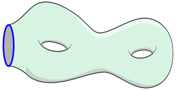


tight pair of pants

kernel is the only part that changes

$$2\frac{\partial}{\partial L_1} L_1$$



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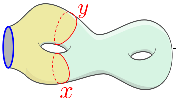

$$+ \cdots$$

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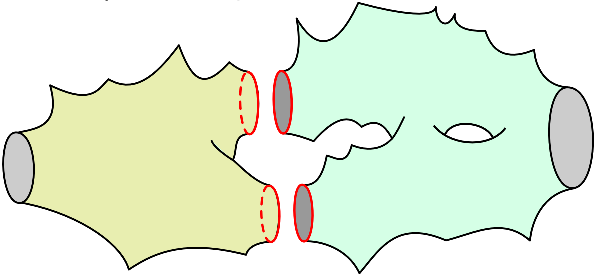

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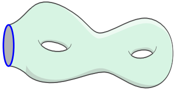

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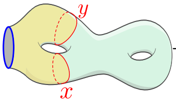

$$+ \cdots$$

Conclusions

- Gravity without coordinates: need to rely on geometric constructions (geodesics, distances, angles, ...) to explore moduli space.
- Bijective approach for maps: often can encode entire geometry in tree data structure, which is much easier to study analytically.
- Extension to hyperbolic geometry procides purely geometric way of computing Weil-Petersson volumes $\longleftrightarrow$ JT gravity path integrals.

Outlook

- Tree bijections open the way to detailed geodesic distance statistics.
- Prospects for 3D gravity?

$$\mathcal{M}_{g,n}^{2+1} = \left\{ \begin{array}{c} \text{solutions to 2+1D GR on } \Sigma_g \times \mathbb{R} \\ \text{coupled to } n \text{ point particles} \end{array} \right\} \approx T\mathcal{M}_{g,n}^{\text{hyp}}$$